Colliding plane waves in general relativity

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PREFACE

For many years after Einstein proposed his general theory of relativity, only a few exact solutions were known. Today the situation is completely different, and we now have a vast number of such solutions. However, very few are well understood in the sense that they can be clearly interpreted as the fields of real physical sources. The obvious exceptions are the Schwarzschild and Kerr solutions. These have been very thoroughly analysed, and clearly describe the gravitational fields surrounding static and rotating black holes respectively.

In practice, one of the great difficulties of relating the particular features of general relativity to real physical problems, arises from the high degree of non-linearity of the field equations. Although the linearized theory has been used in some applications, its use is severely limited. Many of the most interesting properties of space-time, such as the occurrence of singularities, are consequences of the non-linearity of the equations.

In this book we will be considering one of the most obvious situations in which the effects of the non-linearity of Einstein's equations will be manifest. We will be considering the interaction between two waves. By restricting our attention to somewhat idealized situations, it will be possible to describe some types of wave interaction in terms of exact solutions. Moreover, these solutions have a clear physical interpretation in terms of combinations of gravitational or electromagnetic waves and their interaction.

Much attention has been focused on these problems in recent years. An initial approach to the subject was pioneered by Szekeres (1970, 1972) and Khan and Penrose (1971). More recently, an alternative approach using an analogy with stationary axisymmetric solutions has been exploited by Chandrasekhar and Ferrari (1984) and their co-workers.

After spherically symmetric situations, the most studied and best understood space-times are those that are stationary and have axial symmetry. In these situations the field equations can be reduced to a single equation involving a complex potential – the Ernst equation. It is now known that, with this, all possible stationary axisymmetric solutions can be generated in a finite number of steps using standard techniques. In their 1984 paper, Chandrasekhar and Ferrari showed that the main field equations for colliding plane waves can also be written as the same Ernst equation. In fact, it is then found that most of the techniques that have been developed for stationary axisymmetric space-times can also be applied to colliding plane waves. This has introduced considerable mathematical interest in the subject in recent years.

In fact colliding plane wave space-times have been found to have a surprisingly rich structure. Initially, it was widely believed that the collision of plane waves would necessarily produce a future space-like curvature singularity. This seemed to be implied by the focusing properties of plane waves. However, numerous counterexamples have subsequently been produced in which the curvature singularity is replaced by a Killing–Cauchy horizon. Extensions of the space-time through this horizon may, or may not, contain a space-like curvature singularity, or even a time-like curvature singularity which could be avoided by an observer travelling on a time-like world line. Recent research has clarified the singularity structure of most colliding plane wave space-times. These have been found to have a surprisingly rich variation.

In view of the recent advances, it is clearly time to present a comprehensive and unified review of the now vast literature on this topic. The purpose of this book is to provide such a review. Interesting lectures on this topic have been presented by Chandrasekhar (1986) and Ferrari (1989). However, in view of the considerable interest in the subject, a more thorough review is now required.

The first eight chapters of this book cover the background to the subject, presenting the field equations and a discussion of some qualitative aspects of their solution. A detailed discussion of the Khan–Penrose solution is included in this part, since it is the simplest solution and exhibits the general character of most colliding plane wave solutions. Further exact solutions for colliding plane gravitational waves are obtained and described in Chapters 9 to 14. The collision and interaction of electromagnetic waves is then considered in Chapters 15 to 19. The final chapters contain an attempt to summarize all related results for the collision of plane waves of different types and in non-flat backgrounds. A few general conclusions and some outstanding problems that still require attention are also indicated.

In the preparation of this book, I have been greatly assisted by a number of colleagues. I am extremely grateful to Chris Clarke, John Stewart and Sean Hayward for providing me with most helpful comments on the first draft of this work. This final version has been substantially expanded and seems to bear little resemblence to that initial draft. I am also very grateful to Parvinder Singh for reading through a late version, and to Roger Penrose and the American Physical Society for permission to copy Figure 4.1. Finally, I must record my debt to most of the authors of papers on colliding plane waves for regularly sending me preprints of their work prior to publication.

I have also benefited greatly from numerous discussions on colliding wave problems with many colleagues including Professor Chandrasekhar,

Alex Feinstein, Chris Clarke, Sean Hayward, Valeria Ferrari and Basilis Xanthopoulos. The views expressed in the book, however, are my own and I take full responsibility for any errors that it contains.

Loughborough J. B. G.
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1

INTRODUCTION

The subject to be discussed in this book is the collision and interaction of gravitational and electromagnetic waves. This is a particularly important topic in general relativity since the theory predicts that there will be a non-linear interaction between such waves. The effect of the non-linearity, however, is unclear. It is appropriate therefore to look in some detail at the simplest possible situation in which the effect of the non-linearity will be manifest: namely the interaction between colliding plane waves.

1.1 Why consider wave interactions?

In classical theory, Maxwell's equations are linear. An immediate consequence of this is that solutions can be simply superposed. This leads to the prediction that electromagnetic waves pass through each other without any interaction. This prediction is very thoroughly confirmed by observations. Radio waves are transmitted at many different frequencies, yet it is possible for a receiver to select any one particular station and to receive that signal almost exactly as it was transmitted. The only interference that is detected arises from other transmitters using the same frequency, and from the difficulty of isolating just one frequency within the receiver.

After many years' experience, no interaction has ever been detected between propagating electromagnetic waves. This applies not just to radio waves, but to all types of electromagnetic radiation, including light. That we can see clearly through a vacuum, even though light is also passing through it in other directions undetected, is one of the best established of scientific observations. More remarkably, this applies not just to local phenomena, but to the vast regions of space. The light that reaches us from distant galaxies arrives without any apparent interaction with the light that must have crossed its path during the millions of years that it takes to reach us. The apparent linearity of the field equations for electromagnetic waves is thus one of the best established scientific facts.

However, this is not the entire story. Einstein's equations which describe gravitational fields are highly non-linear. It follows that gravitational waves, if they exist, cannot pass through each other without a significant interaction. In this book we will be using the standard general theory of relativity. In this theory gravitational waves are predicted, though their magnitudes are so small that the possibility of detecting them

is only just coming within the scope of the most sophisticated modern apparatus. The purpose of this book is to contribute to an understanding of the character of the interaction that is theoretically predicted between gravitational waves.

In Einstein's theory, gravitational waves are considered as perturbations of space-time curvature that propagate with the speed of light. As these waves pass through each other, theoretically there will be a non-linear interaction through the gravitational field equations. It will be necessary to consider the general character of these interactions, and how the propagating waves are modified by them.

Consider again electromagnetic waves. According to Einstein's theory, all forms of energy have an associated gravitational field. Electromagnetic waves must therefore be coupled to an associated perturbation in the space-time curvature. In the full Einstein–Maxwell theory, Maxwell's equations describing the electromagnetic field remain linear, indicating that there is no direct electromagnetic interaction between waves. However, Einstein's equations, which apply to the gravitational field, are highly non-linear. Thus, as two electromagnetic waves pass through each other, there will be a non-linear interaction between them due to their associated gravitational fields.

This non-linear interaction between electromagnetic waves that is predicted by Einstein's theory must necessarily be very small in order to be consistent with the fact that such interactions have not yet been detected. An interaction, however, is predicted, though its magnitude is likely to be similar to that between gravitational waves.

Since the interactions we will be considering are so weak, it may be considered appropriate initially to use approximation techniques. A number of authors have considered this approach. The modern techniques of numerical relativity have also produced some interesting results. However, these approaches will not be used in this book. The method adopted here will be to concentrate on exact solutions of the Einstein–Maxwell field equations. This has the advantage of being able to clarify something of the global structure of wave interactions. This turns out to be one of their most remarkable features. It leaves us, however, with the problem of finding exact solutions, and these are only possible in a very limited number of situations.

1.2 Simplifying assumptions

The problem that is to be considered in this book is the interaction between two waves. A simple case in which the waves propagate in the same direction has been analysed by Bonnor (1969) and Aichelburg (1971).

They have found that, for the class of vacuum *pp*-waves that will be defined in Section 4.1, the waves can be simply superposed without interaction because of the linearity of the field equations when written in a certain privileged class of coordinate system.

It is therefore appropriate to concentrate on the general case in which the waves propagate in arbitrary different directions. In this case it is always possible to make a Lorentz transformation to a frame of reference in which the waves approach each other from exactly opposite spatial directions. It is therefore only necessary to consider the 'head on' collision between the two waves. However, even this situation is too difficult to analyse without some further simplifying assumptions.

In order to obtain exact solutions, it is appropriate initially to make the additional assumption that the approaching waves have plane symmetry. This is a very severe restriction indeed, even though we intuitively think of plane fronted waves as approximations to spherical waves at large distances from their sources. However, the two cases must be distinguished as their global features are totally different.

The waves we will be considering not only have a plane wave front, but also have infinite extent in all directions in the plane. In contrast, waves generated by finite sources must have curved wave fronts, but it is very difficult to set up boundary conditions and field equations for the interactions between such waves. The reason for concentrating on plane waves is that in this case it is possible to formulate the problem explicitly and to find exact solutions.

In addition to the assumption that the wave front is plane, the imposition of plane symmetry also requires that the magnitude of the wave is constant over the entire plane. Further, it is appropriate to concentrate on 'head on' collisions and thus to impose the condition of global plane symmetry. It is always possible to make a Lorentz transformation to include oblique collisions but the physical interpretation of the solutions is now severely restricted by the above assumptions.

The situation being considered in this book is thus the very restrictive one in which two waves, each with plane symmetry, approach each other from exactly opposite directions. A topic of further research will be to consider how to apply the qualitative results obtained here to more realistic situations, involving waves originating in physical sources. In the absence of more realistic exact solutions, however, the solutions described here form an important first step in an understanding of the non-linear interaction that occurs between waves in Einstein's theory.

2

ELEMENTS OF GENERAL RELATIVITY

It is not the purpose of this chapter to introduce or explain Einstein's general theory of relativity, since the reader who is not already familiar with it is unlikely to gain much from this book. The main purpose here is simply to clarify the notation that will be used. It is also appropriate in this chapter to briefly introduce the Newman–Penrose formalism which facilitates the geometrical analysis of the colliding plane wave problem and which will be used in Chapter 6 to derive the field equations.

2.1 Basic notation

Basically, we will be following a very traditional approach, and the notation adopted will be that of the well known paper of Newman and Penrose (1962).

Accordingly, a space-time will be represented by a connected C^∞ Hausdorff manifold M together with a locally Lorentz metric $g_{\mu\nu}$ with signature $(+,-,-,-)$ and a symmetric linear connection $\Gamma^\lambda{}_{\mu\nu}$. Greek indices are used to indicate the values 0,1,2,3, and the covariant derivative of a vector is given by

$$A^\lambda{}_{;\nu} = A^\lambda{}_{,\nu} + \Gamma^\lambda{}_{\mu\nu} A^\mu \tag{2.1}$$

where a comma denotes a partial derivative.

The curvature tensor is given in terms of the connection by

$$R^\lambda{}_{\kappa\mu\nu} = \Gamma^\lambda{}_{\kappa\nu,\mu} - \Gamma^\lambda{}_{\kappa\mu,\nu} + \Gamma^\lambda{}_{\alpha\mu}\Gamma^\alpha{}_{\kappa\nu} - \Gamma^\lambda{}_{\alpha\nu}\Gamma^\alpha{}_{\kappa\mu}. \tag{2.2}$$

The Ricci tensor, which is the first contraction of the curvature tensor, is given by

$$R_{\mu\nu} = -R^\alpha{}_{\mu\alpha\nu}. \tag{2.3}$$

The curvature tensor has twenty independent components. These can be considered as the ten independent components of the Ricci tensor, and the ten independent components of the Weyl tensor, which is the trace free part of the curvature tensor, and is given by

$$\begin{aligned} C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} &- \tfrac{1}{2}(R_{\lambda\mu}g_{\kappa\nu} - R_{\lambda\nu}g_{\kappa\mu} - R_{\kappa\mu}g_{\lambda\nu} + R_{\kappa\nu}g_{\lambda\mu}) \\ &+ \tfrac{1}{6}R(g_{\lambda\mu}g_{\kappa\nu} - g_{\lambda\nu}g_{\kappa\mu}) \end{aligned} \tag{2.4}$$

where $R = R_\alpha^{\ \alpha}$ is the curvature scalar.

These two groups of components have different physical interpretations. The components of the Ricci tensor are related to the energy-momentum tensor $T_{\mu\nu}$ of the matter field present, through Einstein's equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}. \tag{2.5}$$

These components can be considered to define the amount of curvature that is directly generated by the matter fields that are present at any location. For a vacuum field they will be zero, but they will be non-zero when electromagnetic waves or other fields are present.

The components of the Weyl tensor, on the other hand, define the 'free gravitational field'. They may be considered as describing the components of curvature that are not generated locally. In this sense they describe the pure gravitational field components. They may be interpreted as the components of gravitational waves, or of gravitational fields generated by non-local sources.

2.2 Components of the curvature tensor

It is convenient to represent the curvature tensor in terms of distinct sets of components. Not only may it be divided into the Weyl and Ricci tensors, but each of these tensors may be described in terms of distinct components. The appropriate notation here is that of Newman and Penrose (1962).

It is found to be convenient to introduce a tetrad system of null vectors. These include two real null vectors l^μ and n^μ, a complex null vector m^μ, and its conjugate. They are defined such that their only non-zero inner products are

$$l_\mu n^\mu = 1, \qquad m_\mu \bar{m}^\mu = -1, \tag{2.6}$$

and they must satisfy the completeness relation

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \tag{2.7}$$

Having defined a tetrad basis, the Ricci and Weyl tensors may now be expressed in terms of their tetrad components. The ten independent components of the Ricci tensor can conveniently be divided into a component Λ representing the curvature scalar and the nine independent components of a Hermitian 3×3 matrix Φ_{AB} which represents the trace free part of the Ricci tensor and satisfies

$$\Phi_{AB} = \bar{\Phi}_{BA} \tag{2.8}$$

where A, $B = 0, 1, 2$. These components are defined by

$$
\begin{aligned}
\Phi_{00} &= -\tfrac{1}{2} R_{\mu\nu} l^\mu l^\nu \\
\Phi_{01} &= -\tfrac{1}{2} R_{\mu\nu} l^\mu m^\nu \\
\Phi_{02} &= -\tfrac{1}{2} R_{\mu\nu} m^\mu m^\nu \\
\Phi_{11} &= -\tfrac{1}{4} R_{\mu\nu} (l^\mu n^\nu + m^\mu \bar{m}^\nu) \\
\Phi_{12} &= -\tfrac{1}{2} R_{\mu\nu} n^\mu m^\nu \\
\Phi_{22} &= -\tfrac{1}{2} R_{\mu\nu} n^\mu n^\nu \\
\Lambda &= \tfrac{1}{24} R.
\end{aligned}
\tag{2.9}
$$

The ten independent components of the Weyl tensor, representing the free gravitational field, can more conveniently be expressed as the five complex scalars

$$
\begin{aligned}
\Psi_0 &= -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda l^\mu m^\nu \\
\Psi_1 &= -C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda l^\mu m^\nu \\
\Psi_2 &= -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda \bar{m}^\mu n^\nu \\
\Psi_3 &= -C_{\kappa\lambda\mu\nu} n^\kappa l^\lambda n^\mu \bar{m}^\nu \\
\Psi_4 &= -C_{\kappa\lambda\mu\nu} n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu .
\end{aligned}
\tag{2.10}
$$

These components have distinct physical interpretations that will be mentioned below. They also have particular convenience when considering the algebraic classification of the space-time.

Gravitational fields are usually classified according to the Petrov–Penrose classification of the Weyl tensor. This is based on the number of its distinct principal null directions and the number of times these are repeated. This classification is most conveniently described using a spinor approach. However, there is no need to introduce spinors here, as tetrads are sufficient.

According to the tetrad approach, a null vector k^μ is said to describe a principal null direction of the gravitational field with multiplicity 1, 2, 3 or 4 if it satisfies respectively

$$
\begin{aligned}
k_{[\rho} C_{\kappa]\lambda\mu[\nu} k_{\sigma]} k^\lambda k^\mu &= 0 \\
C_{\kappa\lambda\mu[\nu} k_{\sigma]} k^\lambda k^\mu &= 0 \\
C_{\kappa\lambda\mu[\nu} k_{\sigma]} k^\mu &= 0 \\
C_{\kappa\lambda\mu\nu} k^\mu &= 0
\end{aligned}
\tag{2.11}
$$

where square brackets are used to denote the antisymmetric part. There are at most four principal null directions.

If all four principal null directions are distinct, the space-time is said to be algebraically general, or of type I. If there is a repeated principal null direction, then the space-time is said to be algebraically special. If it has multiplicity two, three or four, the space-time is said to be of types II, III or N respectively. If a space-time has two distinct repeated principal null directions, it is said to be of type D. If the Weyl tensor is zero, the space-time is conformally flat or of type O.

If either of the basis vectors l^μ or n^μ are aligned with principal null directions then either $\Psi_0 = 0$ or $\Psi_4 = 0$ respectively. If the vector l^μ is aligned with the repeated principal null direction of an algebraically special space-time, then $\Psi_0 = \Psi_1 = 0$. If this principal null direction is repeated two, three or four times, then the only non-zero components of the Weyl tensor are the sets (Ψ_2,Ψ_3,Ψ_4), (Ψ_3, Ψ_4) or Ψ_4 respectively. Finally, if l^μ and n^μ are both aligned with the distinct principal null directions of a type D space-time, then the only non-zero component of the Weyl tensor is Ψ_2.

The physical meaning of the different components of the Weyl tensor has been investigated by Szekeres (1965), and may be summarized as follows:

Ψ_0 denotes a transverse wave component in the n^μ direction.
Ψ_1 denotes a longitudinal wave component in the n^μ direction.
Ψ_2 denotes a coulomb component.
Ψ_3 denotes a longitudinal wave component in the l^μ direction.
Ψ_4 denotes a transverse wave component in the l^μ direction.

This interpretation will be very useful when we come to analyse the interaction between two gravitational waves. It is possible to align the two basis vectors l^μ and n^μ with the two waves. The interaction to be considered here is between transverse waves, so the problem is to find the interaction between the Ψ_4 and Ψ_0 components.

2.3 Spin coefficients

In some situations it has been found convenient to modify the notation for tetrads, spinors and spin coefficients in recent years.[1] However, it is most convenient here to continue to use the original notation of Newman and Penrose (1962).[2] Those not familiar with this notation need not be

[1] See Geroch, Held and Penrose (1973), Penrose and Rindler (1985).

[2] For a detailed introduction to the Newman–Penrose formalism see Pirani (1965), Carmeli (1977), Alekseev and Klebnikov (1978), Frolov (1979) and Kramer *et al.* (1980).

too alarmed, as we have no need here to introduce spinors or their more intricate properties. Effectively, we are only introducing the notation because of its convenience for describing certain geometrical properties of the solutions.

The main feature of the Newman–Penrose formalism is the introduction of spin coefficients. These are complex linear combinations of the Ricci rotation coefficients associated with the null tetrad. They are defined by

$$\begin{aligned}
\kappa &= l_{\mu;\nu} m^\mu l^\nu & \nu &= -n_{\mu;\nu} \bar{m}^\mu n^\nu \\
\rho &= l_{\mu;\nu} m^\mu \bar{m}^\nu & \mu &= -n_{\mu;\nu} \bar{m}^\mu m^\nu \\
\sigma &= l_{\mu;\nu} m^\mu m^\nu & \lambda &= -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu \\
\tau &= l_{\mu;\nu} m^\mu n^\nu & \pi &= -n_{\mu;\nu} \bar{m}^\mu l^\nu \\
\epsilon &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu) \\
\alpha &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu) \\
\beta &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu) \\
\gamma &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu).
\end{aligned} \tag{2.12}$$

The spin coefficients have the following geometrical interpretations. If $\kappa = 0$, then l^μ is tangent to a geodesic null congruence. If, in addition, $\mathcal{R}e\ \epsilon = 0$, then l^μ is the tangent vector corresponding to an affine parametrization, and $-\mathcal{R}e\ \rho$, $\mathcal{I}m\ \rho$ and $|\sigma|$ define the expansion, twist and shear of the congruence respectively. Also, $\arg\sigma$ determines the shear axes. With a change of sign, a positive value for $\mathcal{R}e\ \rho$ is more appropriately referred to as the contraction of the congruence.

It may also be noted that l_μ is propertional to the gradient of a scalar field if, and only if, it is tangent to a twist-free null geodesic congruence ($\kappa = 0$, $\rho - \bar\rho = 0$). Also, l_μ is equal to the gradient of a scalar field when $\kappa = 0$, $\rho - \bar\rho = 0$, $\epsilon + \bar\epsilon = 0$ and $\bar\alpha + \beta = \tau$.

In the congruence defined by n^μ the coefficients $-\nu$, $-\gamma$, $-\mu$, $-\lambda$ correspond to κ, ϵ, ρ, σ respectively. The geometrical properties of the solutions given later can be very conveniently analysed in terms of these spin coefficients.

We also need to define the intrinsic derivatives. These are directional derivatives in the directions of the four tetrad vectors, and are defined by

$$D = l^\mu \nabla_\mu, \quad \Delta = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \bar\delta = \bar{m}^\mu \nabla_\mu \tag{2.13}$$

where ∇_μ is the covariant derivative operator, previously denoted by a semicolon.

Basic to their formalism is the set of Newman–Penrose identities. The first group of these are linear combinations of the Ricci identities applied to the tetrad vectors. These will only be referred to occasionally in the following sections, but it is still worth quoting them here as follows:

$$D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \tag{2.14a}$$

$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho}) + \sigma(3\epsilon - \bar{\epsilon}) - \kappa(\pi - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0 \tag{2.14b}$$

$$\begin{aligned} D\tau - \Delta\kappa = {}& \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) \\ & + \Psi_1 + \Phi_{01} \end{aligned} \tag{2.14c}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\rho + \epsilon) + \Phi_{10} \tag{2.14d}$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \tag{2.14e}$$

$$\begin{aligned} D\gamma - \Delta\epsilon = {}& \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa \\ & + \Psi_2 - \Lambda + \Phi_{11} \end{aligned} \tag{2.14f}$$

$$D\lambda - \bar{\delta}\pi = \rho\lambda + \bar{\sigma}\mu + \pi^2 + \pi(\alpha - \bar{\beta}) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20} \tag{2.14g}$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \sigma\lambda + \pi\bar{\pi} - \mu(\epsilon + \bar{\epsilon}) - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_2 + 2\Lambda \tag{2.14h}$$

$$\begin{aligned} D\nu - \Delta\pi = {}& \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) \\ & + \Psi_3 + \Phi_{21} \end{aligned} \tag{2.14i}$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu}) - \lambda(3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4 \tag{2.14j}$$

$$\begin{aligned} \delta\rho - \bar{\delta}\sigma = {}& \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) \\ & - \Psi_1 + \Phi_{01} \end{aligned} \tag{2.14k}$$

$$\begin{aligned} \delta\alpha - \bar{\delta}\beta = {}& \rho\mu - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) \\ & - \Psi_2 + \Lambda + \Phi_{11} \end{aligned} \tag{2.14l}$$

$$\begin{aligned} \delta\lambda - \bar{\delta}\mu = {}& \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) \\ & - \Psi_3 + \Phi_{21} \end{aligned} \tag{2.14m}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22} \tag{2.14n}$$

$$\delta\gamma - \Delta\beta = \tau(\mu + \gamma) - \gamma\bar{\alpha} - \sigma\nu - \epsilon\bar{\nu} - \beta(2\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12} \tag{2.14o}$$

$$\delta\tau - \Delta\sigma = \mu\sigma + \rho\bar{\lambda} + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \kappa\bar{\nu} + \Phi_{02} \tag{2.14p}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \kappa\nu - \Psi_2 - 2\Lambda \tag{2.14q}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3. \tag{2.14r}$$

We will also require the commutation relations between the intrinsic derivatives. When applied to scalar functions, these are given by

$$\begin{aligned} \Delta D - D\Delta &= (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta \\ \delta D - D\delta &= (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta \\ \delta\Delta - \Delta\delta &= -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta \\ \bar{\delta}\delta - \delta\bar{\delta} &= (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\beta - \bar{\alpha})\bar{\delta} + (\alpha - \bar{\beta})\delta. \end{aligned} \tag{2.15}$$

These can be applied to the coordinates to give the so-called metric equations, and also to the spin coefficients and the curvature tensor components.

An essential part of the general Newman–Penrose formalism is the set of Bianchi identies. These, however, will not be used explicitly in this text and therefore do not need to be repeated here.

2.4 Einstein–Maxwell fields

In this formalism, Einstein's field equations are applied simply by replacing the expressions for the Ricci tensor components in the above identities by the appropriate components of the energy-momentum tensor according to equation (2.5). For an electromagnetic field in a vacuum, for example, it is convenient to represent the electromagnetic field tensor $F^{\mu\nu}$ by three complex scalars defined by

$$\begin{aligned} \Phi_0 &= F_{\mu\nu} l^\mu m^\nu \\ \Phi_1 &= \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) \\ \Phi_2 &= F_{\mu\nu} \bar{m}^\mu n^\nu . \end{aligned} \tag{2.16}$$

It is also possible to scale the electromagnetic field tensor such that Einstein's gravitational field equations are then given by

$$\Phi_{AB} = \Phi_A \bar{\Phi}_B, \qquad \Lambda = 0, \tag{2.17}$$

and, in this case, Maxwell's equations take the form

$$\begin{aligned} D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \\ D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \\ \delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \\ \delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 \end{aligned} \tag{2.18}$$

It is also possible to classify the electromagnetic field in a similar way to the classification of the Weyl tensor. An electromagnetic field is said to be non-null or null if it has two distinct or one repeated principal null direction k^μ satisfying respectively

$$\begin{aligned} &F_{\mu[\nu} k_{\lambda]} k^\mu = 0 \\ &F_{\mu\nu} k^\mu = 0, \qquad F_{[\mu\nu} k_{\lambda]} = 0. \end{aligned} \tag{2.19}$$

Aligning l^μ with a principal null direction of the electromagnetic field makes $\Phi_0 = 0$. If l^μ is a repeated principal null direction, then $\Phi_0 = \Phi_1 = 0$, and the only non-zero component of the null field is Φ_2.

3

COLLIDING IMPULSIVE GRAVITATIONAL WAVES

One of the first exact solutions of Einstein's equations describing the mutual scattering of gravitational waves was that of Khan and Penrose (1971). This describes the collision and subsequent interaction of two plane impulsive gravitational waves and their subsequent interaction. Following this pioneering work, many more exact solutions have been obtained. However, the general character of almost all solutions that have subsequently been found is basically the same as that of the original solution of Khan and Penrose. It is therefore appropriate to devote a chapter, at this early stage in a discussion of the subject, to this particular solution. The aim is to describe it in detail, pointing out some of its properties and the problems that it highlights, to which further attention will be given in subsequent chapters.

3.1 The approaching waves

In the Khan–Penrose solution the approaching waves are plane impulsive gravitational waves. Metrics describing such waves are well known. In a flat background, one of the approaching waves may conveniently be described by the Brinkmann–Peres–Takeno line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + \delta(u)(X^2 - Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{3.1}$$

where $\delta(u)$ is the Dirac delta function, here representing the impulsive wave component. Although the metric in this case is distribution valued, it will be shown later that it does in fact give rise to a geometrically acceptable space-time.

The line element (3.1) involves a null coordinate u, and the plane impulsive wave occurs on the null hypersurface $u = 0$. The opposing wave may also initially be described by the same line element, but with the null coordinate u replaced by another null coordinate v. Naturally, another space-like coordinate must replace r, but for 'head on' collisions the other space-like coordinates X and Y may be retained provided that the two approaching waves have their polarization vectors aligned.

It is found to be most convenient to describe the entire space-time using the two null coordinates u and v. The initial wave (3.1) can be expressed in terms of these two coordinates by using the transformation

$$\begin{aligned} u &= u \\ r &= v - \tfrac{1}{2}\Theta(u)(1-u)x^2 + \tfrac{1}{2}\Theta(u)(1+u)y^2 \\ X &= \big(1 - u\Theta(u)\big)x \\ Y &= \big(1 + u\Theta(u)\big)y \end{aligned} \tag{3.2}$$

where $\Theta(u)$ is the Heaviside step function. It may be noted that the discontinuity in this transformation is related to the discontinuity in the line element (3.1). At this stage, this transformation may be considered as a purely symbolic manipulation which will later be shown to be geometrically acceptable. Using the transformation (3.2), the line element (3.1) takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(1 - u\Theta(u)\big)^2\mathrm{d}x^2 - \big(1 + u\Theta(u)\big)^2\mathrm{d}y^2. \tag{3.3}$$

The gravitational wave is here described by the single component

$$\Psi_4 = \delta(u). \tag{3.4}$$

Strictly, the line element (3.3) with wave component (3.4) only describes one wave as it approaches the other. It therefore only applies to the region $v < 0$. The opposing wave, prior to the collision, is described by the same line element but with u and v interchanged. It is thus described by the component

$$\Psi_0 = \delta(v). \tag{3.5}$$

Since this also only describes the approaching wave, it similarly is defined only in the region $u < 0$. The initial situation of two approaching waves has now been defined. The question to be discussed is what happens to the two waves when they meet and pass through each other. How do they interact gravitationally?

At this point it is convenient to divide space-time up into four distinct regions, as illustrated in Figure 3.1. Region I, where $u < 0$ and $v < 0$, is the flat background having line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{3.6}$$

Region II, where $u \geq 0$ and $v < 0$, contains the wave with line element (3.3)

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1-u)^2\mathrm{d}x^2 - (1+u)^2\mathrm{d}y^2. \tag{3.7}$$

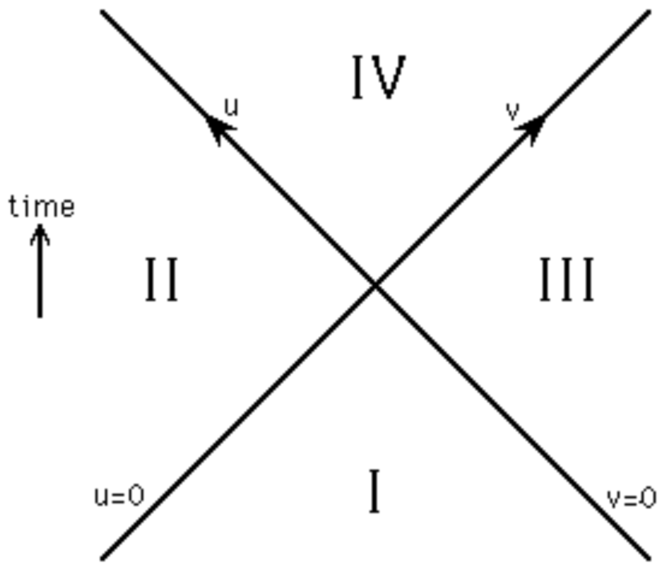


Figure 3.1 Space-time is conveniently divided into four regions as shown. Two space-like coordinates have been suppressed. Region I is the background, regions II and III contain the approaching waves, and region IV is the interaction region following the collision at the point $u = 0$, $v = 0$.

Region III, where $v \geq 0$ and $u < 0$, contains the opposing wave, which is described by the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1 - v)^2\mathrm{d}x^2 - (1 + v)^2\mathrm{d}y^2. \qquad (3.8)$$

Region IV is the interaction region $u \geq 0$ and $v \geq 0$. Its line element will have to be found as a solution of Einstein's equations, subject to appropriate boundary conditions set on the two null surfaces $v = 0$, $u \geq 0$ and $u = 0$, $v \geq 0$. In this case initial data is well set, so a unique solution exists in this region.

Before presenting the solution for region IV, it is worth noticing that the metrics (3.7) and (3.8) for regions II and III are singular on the null hypersurfaces $u = 1$ and $v = 1$ respectively. Such singularities do not occur in the line elements of the form (3.1). It is therefore tempting to regard them simply as coordinate singularities, since they can be removed by transformations of the type (3.2). In fact these singularities do seem to have a particular significance, and they will be discussed in detail in the following sections.

3.2 The solution describing the interaction

The field equations for the interaction region IV, and the appropriate boundary conditions will be discussed in full in the following chapters.

At this point it is only appropriate to state the resulting line element as given by Khan and Penrose (1971). This has the form

$$
\begin{aligned}
\mathrm{d}s^2 = 2&\frac{(1-u^2-v^2)^{3/2}}{\sqrt{1-u^2}\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})^2}\mathrm{d}u\mathrm{d}v \\
&-(1-u^2-v^2)\bigg(\frac{(1-u\sqrt{1-v^2}-v\sqrt{1-u^2})}{(1+u\sqrt{1-v^2}+v\sqrt{1-u^2})}\mathrm{d}x^2 \\
&\qquad+\frac{(1+u\sqrt{1-v^2}+v\sqrt{1-u^2})}{(1-u\sqrt{1-v^2}-v\sqrt{1-u^2})}\mathrm{d}y^2\bigg).
\end{aligned}
\tag{3.9}
$$

Using a notation that will be defined later, the gravitational field can be described by the components

$$
\begin{aligned}
\Psi_0^\circ &= \frac{1}{\sqrt{1-u^2}}\delta(v) + \frac{3u\sqrt{1-u^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})}{(1-v^2)(1-u^2-v^2)^2} \\
\Psi_2^\circ &= \frac{(uv+\sqrt{1-u^2}\sqrt{1-v^2})^2}{\sqrt{1-u^2}\sqrt{1-v^2}(1-u^2-v^2)^2} - \frac{uv}{(1-u^2-v^2)^2} \\
\Psi_4^\circ &= \frac{1}{\sqrt{1-v^2}}\delta(u) + \frac{3v\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})}{(1-u^2)(1-u^2-v^2)^2}.
\end{aligned}
\tag{3.10}
$$

It may be noticed that the impulsive wave components continue after the point of collision at $u=0$, $v=0$, but with a scaled magnitude. Both wave components also develop tails, so that the components Ψ_0 and Ψ_4 are non-zero throughout the interaction region. The two waves may therefore be considered to scatter each other. Another most significant feature that is demonstrated by the components (3.10) is that the component Ψ_2 also becomes non-zero. This is in fact a familiar feature of interacting waves. In fact it will be shown later that it is not possible for a vacuum space-time to contain only the components Ψ_0 and Ψ_4.

It should also be noticed that all the components described by (3.10) become unbounded on the hypersurface $u^2+v^2=1$ on which the line element (3.9) is singular. That this corresponds to a scalar polynomial curvature singularity can be seen by considering the scalar invariant

$$
3{\Psi_2}^2 + \Psi_0\Psi_4 = \delta(u)\delta(v) + \frac{3S^4(S^4+S^2P+P^2)}{(1-u^2-v^2)^7}
\tag{3.11}
$$

where $S = uv+\sqrt{1-u^2}\sqrt{1-v^2}$, and $P = uv\sqrt{1-u^2}\sqrt{1-v^2}$.

3.3 The structure of the solution

The most important feature illustrated by this solution is the existence of the future space-like curvature singularity at $u^2 + v^2 = 1$. It may be noticed that this is connected at the boundaries to the coordinate singularities in regions II and III. If these coordinate singularities are considered as having no physical significance, then the naive structure of the solution is as described in Figure 3.2.

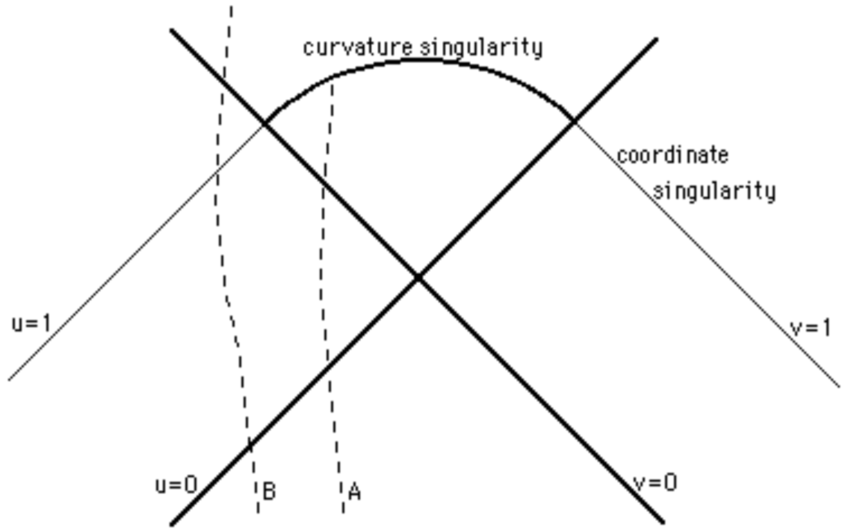


Figure 3.2 The apparent singularity structure of the Khan–Penrose solution. There is a curvature singularity in region IV which is attached to coordinate singularities in regions II and III. Apparent worldlines of particles A and B are referred to in the text.

Refering to this figure, we may consider the worldline of a test particle A, which first encounters one gravitational wave, and then the other, and must subsequently fall into a curvature singularity after a finite proper time. This would seem to be the fate of all objects that detect two plane gravitational waves. Such situations are not unfamiliar in general relativity. They occur for example in cosmology if the cosmic expansion ceases. The present situation may be similar, in that once a general contraction is established, a future singularity becomes inevitable. The possibility of such an interpretation is at least a little comforting, in that it suggests that the singularity may be removed to a cosmological time scale in the future. This will need to be considered. We will clearly also have to spend some time trying to establish whether or not such singularities are a generic feature of colliding waves. In fact it will be shown in subsequent chapters that curvature singularities of this type are a general feature of colliding wave solutions whenever the initial data has global plane

symmetry, and the background in region I is flat. However, there is a large class of notable exceptions in which this curvature singularity is replaced by a Cauchy horizon.

It may be pointed out at this stage, however, that the time between the collision and the subsequent singularity is inversely proportional to the strength of the waves. This can be seen simply by replacing u by au and v by bv in the above solution. In this case, the two approaching waves prior to the collision are given by $\Psi_4 = a\delta(u)$ and $\Psi_0 = b\delta(v)$. The singularity then occurs on the space-like surface given by $1 - a^2u^2 - b^2v^2 = 0$.

Refering again to Figure 3.2, consider the worldline of the test particle B. This encounters a gravitational wave and, after some time, appears to encounter the coordinate singularity in region II. If we consider this to be merely a coordinate singularity, then the particle may simply be able to pass through it. It would then subsequently meet the opposing wave and, having passed through this, it would then continue indefinitely into the future. However, once the particle has passed the second gravitational wave, it will be able to look back and see a naked curvature singularity. Such possibilities must at least be considered with suspicion in view of the cosmic censorship hypothesis suggested by Penrose. It will therefore be necessary to analyse in detail the character of the singularities in regions II and III. In fact it will be shown that paths of the type described are not permitted. It will be argued that pictures such as Figures 3.1 and 3.2 can be misleading in that they only illustrate the time-like hypersurface $x = y = 0$, and do not indicate the causal structure of the complete space-time.

4

PLANE WAVES

The purpose of this chapter is to describe in some detail the character of the plane waves whose interactions we will be considering in the subsequent chapters.

4.1 The class of *pp*-waves

It is first convenient to introduce the widely known class of *pp*-waves. These are plane-fronted gravitational waves with parallel rays. They are defined by the property that they admit a covariantly constant null vector field. It is possible to interpret such a field as the rays of gravitational or other null waves.

It is possible to identify the tetrad vector l^μ with this field. It can then be seen from (2.12) that the defining property that $l_{\mu;\nu} = 0$, among other conditions, immediately implies that $\rho = \sigma = \kappa = 0$. It follows that this vector field is tangent to a non-expanding, shear-free and twist-free null geodesic congruence. Since the congruence is twist-free, there exists a family of 2-surfaces orthogonal to l^μ that may be considered as wave surfaces (Kundt, 1961). Moreover, since the congruence is expansion-free, the wave surfaces are plane and since, in addition, $\tau = 0$, the rays orthogonal to the wave surfaces are parallel.

This class of solutions was first discovered by Brinkmann (1923), and subsequently rediscovered by several authors, for example Peres (1959). Using a null coordinate u, defined such that $l_\mu = u_{,\mu}$, the metric can be written in the Kerr–Schild form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + H(u, X, Y)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.1}$$

where the coordinates X and Y span the wave surfaces. These metrics are either of algebraic type N, or are conformally flat. The only non-zero components of the curvature tensor are given by

$$\begin{aligned} \Phi_{22} &= \tfrac{1}{4}(H_{,XX} + H_{,YY}) \\ \Psi_4 &= \tfrac{1}{4}(H_{,XX} - H_{,YY} + 2iH_{,XY}). \end{aligned} \tag{4.2}$$

For reviews of these solutions see Takeno (1961), Ehlers and Kundt (1962), Zakharov (1973), and Kramer *et al.* (1980).

The gravitational wave can be expressed in terms of the single component $\Psi_4 = Ae^{i\alpha}$. It is then possible to use the analogy between the Weyl tensor component Ψ_4 and the electromagnetic field tensor component Φ_2, and to regard A as the amplitude of the gravitational wave and α as its polarization. In this form, the vacuum *pp*-waves for which α is constant are said to be linearly polarized.

It may also be noticed at this point that the vacuum field equations for space-times with the line element (4.1) reduce to the two-dimensional Laplace equation

$$H_{,XX} + H_{,YY} = 0. \tag{4.3}$$

Since this equation is linear, it follows that distinct solutions with different expressions for H may be simply superposed. It follows that independent gravitational waves of this type which propagate in the same direction do not interact. This was first pointed out by Bonnor (1969) and Aichelburg (1971).

There is another interesting result that has been proved by Yurtsever (1988*b*) which relates to the class of so-called 'sandwich waves'. This states that any gravitational wave space-time that is flat before the arrival of the wave and returns to perfect flatness after the wave passes is necessarily a *pp*-wave space-time.

4.2 The class of plane waves

In Einstein–Maxwell theory, the particular class of *plane waves* are defined to be *pp*-waves in which the field components are the same at every point of the wave surfaces. This is the sense in which they are said to have 'plane symmetry'. This class of solutions was first considered by Baldwin and Jeffery (1926).

Using the above notation, this condition requires that Ψ_4 and Φ_2 are independent of the space-like coordinates X and Y which span the wave surfaces. The line element for a plane wave can thus be written in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + (h_{11}X^2 + 2h_{12}XY + h_{22}Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.4}$$

where h_{ij} are functions of u only. Terms in H that are linear in X and Y have been removed by a simple coordinate transformation. The non-zero curvature tensor components are then

$$\begin{aligned} \Phi_{22} &= \tfrac{1}{2}(h_{11} + h_{22}) \\ \Psi_4 &= \tfrac{1}{2}(h_{11} - h_{22} + 2ih_{12}). \end{aligned} \tag{4.5}$$

The line element (4.4) describes a vacuum space-time, in this case a pure gravitational plane wave, if $h_{22} = -h_{11}$. The gravitational wave will have constant linear polarization if, in addition, h_{12} is proportional to h_{11}. In this case the metric function H can be expressed in the form

$$H = h(u)\big(\cos\alpha(X^2 - Y^2) + 2\sin\alpha\, XY\big) \tag{4.6}$$

where $h(u)$ is an arbitrary function and α is the (constant) polarization of the wave and $\Psi_4 = h(u)e^{i\alpha}$.

It is sometimes convenient to introduce the concept of a polarization vector of the gravitational plane wave. Describing the space-time using the above notation, this may be considered to be a vector in the space-like wave surfaces that is inclined at the angle α to the X-coordinate direction.

For a gravitational wave with constant linear polarization it is possible to rotate the coordinates such that $h_{12} = 0$, or $\alpha = 0$. In this case the polarization vector is aligned with the X-coordinate direction, and the line element is then

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + h_{11}(u)(X^2 - Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.7}$$

and $\Psi_4 = h_{11}(u)$. For a general plane gravitational wave, however, the function h_{12} will not be proportional to h_{11}, and the polarization will then be variable.

On the other hand, it can be seen that (4.4) describes a pure electromagnetic wave with zero Weyl tensor if $h_{22} = h_{11}$ and $h_{12} = 0$. It then takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + h_{11}(u)(X^2 + Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.8}$$

where $\Phi_{22} = h_{11}(u)$.

In order to consider the collision and interaction of plane waves of the above type, it will be found convenient to transform the general line element (4.4) to a form that was first considered by Rosen (1937). This uses two null coordinates u and v, and can be obtained from (4.4) using the transformation

$$\begin{aligned} X &= ax + by \\ Y &= ex + cy \\ r &= v + \tfrac{1}{2}(aa' + ee')x^2 + \tfrac{1}{2}(ba' + ab' + ec' + ce')xy + \tfrac{1}{2}(bb' + cc')y^2 \end{aligned} \tag{4.9}$$

where the coefficients a, b, c and e are all functions of u only, which are

constrained by the equations

$$\begin{aligned} &a'' + h_{11}a + h_{12}e = 0 \\ &b'' + h_{11}b + h_{12}c = 0 \\ &c'' + h_{12}b + h_{22}c = 0 \\ &e'' + h_{12}a + h_{22}e = 0 \\ &ba' - ab' - ec' + ce' = 0. \end{aligned} \tag{4.10}$$

This transformation produces the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (a^2 + e^2)\mathrm{d}x^2 - 2(ab + ce)\mathrm{d}x\mathrm{d}y - (b^2 + c^2)\mathrm{d}y^2 \tag{4.11}$$

in which the coefficients are functions of u only. The curvature tensor components are now

$$\begin{aligned} \Phi_{22} &= -\frac{ca'' + ac'' - eb'' - be''}{2(ac - be)} \\ \Psi_4 &= -\frac{(ca'' - ac'' - eb'' + be'') - i(ba'' - ab'' + ec'' - ce'')}{2(ac - be)}. \end{aligned} \tag{4.12}$$

If the gravitational wave component has constant linear polarization, h_{12} in (4.4) may be put equal to zero, and b and e may also be taken as zero, producing a significant simplification.

It may be observed that there is a redundant function in the line element (4.11). It is therefore appropriate to rewrite it in an alternative form. For later convenience we choose the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - e^{-U}(e^{V}\cosh W\mathrm{d}x^2 - 2\sinh W\mathrm{d}x\mathrm{d}y + e^{-V}\cosh W\mathrm{d}y^2) \tag{4.13}$$

where U, V and W are functions of u only. In this form

$$\begin{aligned} \Phi_{22} &= \tfrac{1}{4}(2U_{uu} - U_u^2 - W_u^2 - V_u^2\cosh^2 W) \\ \Psi_4 &= -\tfrac{1}{2}(V_{uu}\cosh W - V_u U_u\cosh W + 2V_u W_u\sinh W) \\ &\qquad - \tfrac{1}{2}i(W_{uu} - W_u U_u - V_u^2\cosh W\sinh W). \end{aligned} \tag{4.14}$$

In the case when the gravitational wave component has linear polarization, it is always possible to put $W = 0$.

That the above space-time describes plane waves has been demonstrated by Bondi, Pirani and Robinson (1959). They have shown that this class of solutions has plane symmetry and contains waves that propagate along the null hypersurfaces given by $u =$ constant. They have

demonstrated that relative accelerations are produced in test particles when such a wave passes through them. They have also shown that the metric given by (4.13) admits a 5-parameter group of motions that are generated by the Killing vectors

$$\begin{aligned} \xi_1 &= \partial_x \\ \xi_2 &= \partial_y \\ \xi_3 &= \partial_v \\ \xi_4 &= x\partial_v + P_-(u)\partial_x + N(u)\partial_y \\ \xi_5 &= y\partial_v + P_+(u)\partial_y + N(u)\partial_x \end{aligned} \tag{4.15}$$

where

$$P_\pm(u) = \int e^{U\pm V}\cosh W \mathrm{d}u, \qquad N(u) = \int e^U \sinh W \mathrm{d}u. \tag{4.16}$$

All the corresponding operators commute except for $[\xi_1, \xi_4] = \xi_3$ and $[\xi_2, \xi_5] = \xi_3$. These commutators indicate that the structure constants for plane gravitational waves are the same as those for plane electromagnetic waves in a flat space-time. This analogy with plane electromagnetic waves further justifies their interpretation as plane waves.

4.3 Particular cases

In the above discussion, some attention has already been paid to the cases of pure electromagnetic waves, and pure gravitational waves with linear polarization. These have been described by the line elements (4.8) and (4.7), but the profile of the waves, which is determined by the arbitrary function $h_{11}(u)$, has been left totally general. In this section, some waves with specific profiles will be considered in a little more detail. The purpose here is to describe some of the properties of the waves that will be considered in the following chapters.

Consider first a shock electromagnetic wave with a step profile

$$\Phi_{22} = a^2\Theta(u). \tag{4.17}$$

For this the line element (4.8) becomes

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a^2\Theta(u)(X^2+Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.18}$$

In transforming this to Rosen form, the region in front of the wave is flat and, for $u<0$, has the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{4.19}$$

In the region $u \geq 0$, which contains the electromagnetic wave, the line element is

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 au(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{4.20}$$

Notice that, in this form, the metric is differentiable across the wave front $u = 0$.

Consider similarly a gravitational step wave with

$$\Psi_4 = a^2\Theta(u). \tag{4.21}$$

The line element (4.7) becomes

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a^2\Theta(u)(X^2 - Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2. \tag{4.22}$$

The region in front of the wave is flat having the line element (4.19), but for $u \geq 0$ we obtain

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 au\,\mathrm{d}x^2 - \cosh^2 au\,\mathrm{d}y^2. \tag{4.23}$$

Finally, we may point out that the case of an impulsive gravitational wave has already been described in Section 3.1. If $\Psi_4 = a\delta(u)$, the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a\delta(u)(X^2 - Y^2)\mathrm{d}v^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.24}$$

can be transformed to

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1 - au)^2\mathrm{d}x^2 - (1 + au)^2\mathrm{d}y^2 \tag{4.25}$$

in the region following the wave front. For $u < 0$, the line element is (4.19) as before.

This example of an impulsive gravitational wave is particularly important since an impulsive wave can be considered in some sense as an idealization of any wave of finite duration. It may be noticed that the metric (4.25) is actually flat. The curvature components occur only on the wave front $u = 0$. Thus, this line element can also be used, with a possible linear transformation of the coordinate u, to describe the region following any sandwich gravitational wave with constant linear polarization.

4.4 Global properties

It can be seen that the general line element (4.4), including the particular cases (4.18), (4.22) and (4.24), describes a space-time containing plane waves. The same form of the metric can also describe a sandwich wave

and the regions both in front of and behind it. The wave, however, is of infinite extent in all directions in its plane. The energy of the wave is therefore, in a global sense, infinite.

One further problem of this Brinkmann form of the metric is that discontinuities arise in the metric components. This can be seen explicitly in (4.18), (4.22) and (4.24). However, this difficulty can easily be removed since it is possible to transform these line elements to the Rosen form (4.11) or (4.13), which is always continuous. The discontinuities in the Brinkmann form of the metric arise from the particular choice of coordinates.

On the other hand, it can also be seen that the line elements in Rosen form (4.20), (4.23) and (4.25) are singular. They all have coordinate singularities on some hypersurface behind the wave. As will be described in the following chapter, this can be interpreted in terms of the focusing of the null congruences opposing the waves that are associated with the coordinate v. For single plane waves, these are only coordinate singularities. In fact, coordinate singularities of various kinds inevitably arise whenever the metric is taken in Rosen form (4.11) or (4.13).

The focusing property of plane waves has been further analysed by Bondi and Pirani (1989) and described in terms of caustics. They have considered the effect of a sandwich plane gravitational wave on a set of test particles that are strung out in a fixed direction, which depends on the polarization of the wave, and are initially at rest in a flat space-time. They have proved, at least for waves that have constant linear polarization, that all the particles will collide after a finite time independent of how far apart they were initially. The occurrence of such caustics has been described in detail. They are associated with the coordinate singularities that occur in the Rosen form of the metric. For pure electromagnetic waves, a whole three-dimensional set of basic world lines may pass through a single event, as also pointed out by Cantoni (1971). This is one aspect of the focusing power of gravitational and electromagnetic waves. Further aspects will be discussed in the following chapters.

Bondi and Pirani (1989) have also considered further geometrical properties of plane waves as seen by an observer who passes through the wave. Since the world lines of initially stationary test particles may meet a finite time after passing through a wave, however far apart they were initially, the observable pasts of both particles will eventually coincide. It follows that, after passing a wave front, an observer will, within a finite time, have seen the whole of an infinite spatial volume in a strip of the half hyperplane in front of the wave.

Further properties of plane waves have been described by Penrose (1965*a*) from a different point of view. He has considered the structure

of the future light cone of a point Q in front of the wave. As this cone expands, part of it crosses the wave and is distorted by it. Eventually it will encounter the coordinate singularity behind the wave. The light emitted from Q that passes through the wave will be focused by it. This process will be described in the next chapter. For a gravitational wave the light will be focused to a line, but for a pure electromagnetic wave the light will be focused to a point. In either case the light will appear on the past light cone of this line or point, which may be denoted by R. It can thus be seen that the future light cone of a point Q in front of the wave is identical to the past light cone of the "point" R behind the wave. This is illustrated in Figure 4.1.

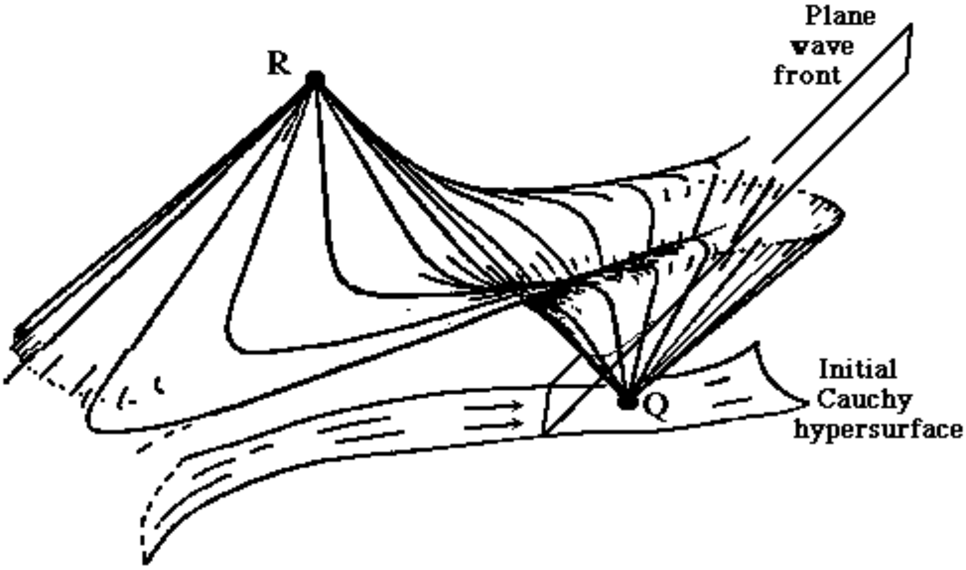


Figure 4.1 (Penrose 1965*a*) The future light cone of the point Q is distorted as it passes through a plane wave and is again focused to another vertex R which may be a point or a line (one spatial dimension has been suppressed).

Another remarkable property of plane waves can be deduced immediately from Figure 4.1, as has been argued in detail by Penrose (1965*a*). It implies that a plane wave space-time contains no global Cauchy hypersurface. In other words, it implies that it is not possible to set up initial values for a plane wave on any global space-like hypersurface. Such a hypersurface must lie entirely in the past of any future null cone from any point on the surface. However, in this case, the future null cone of Q folds down to focus again at R. The Cauchy hypersurface passing through Q must therefore lie entirely below the past null cone of R. It can not, therefore, extend to spatial infinity behind the wave. Thus no space-like hypersurface exists that is adequate for the global specification of Cauchy

data.

It also follows from the above discussion that it is not possible to embed a plane wave globally in any hyperbolic normal pseudo-Euclidean space.

It may be noted that a further mathematical description of the geometrical properties of plane waves, particularly related to colliding wave problems, has also been given by Yurtsever (1988*b*). These properties will be also described in more detail in some of the following chapters.

5

GEOMETRICAL CONSIDERATIONS

The purpose of this chapter is to consider in more detail some of the geometrical properties of space-times that contain plane waves. Part of the discussion here follows naturally from that at the end of the previous chapter. Here, some more general results of a geometrical nature will be considered, and an intuitive approach to the collision of plane waves will also be developed.

5.1 The focusing of congruences

In a general space-time, consider a family of null geodesics. The tetrad vector l^μ may be aligned with the tangent vector field of this congruence. It is also possible to choose an affine parameter along the congruence. In this case, the spin coefficients κ and ϵ are zero, and the first two of the Newman–Penrose equations (2.14*a,b*) are

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \qquad (5.1a)$$
$$D\sigma = \sigma(\rho + \bar{\rho}) + \Psi_0 \qquad (5.1b)$$

where D is the directional derivative along the congruence. Assume initially that the congruence starts in a vacuum region of space-time with Φ_{00} and Ψ_0 both zero, and that the geodesics are parallel having zero contraction, twist and shear, so that ρ and σ are both zero. Equations (5.1) are identically satisfied.

Now assume that this congruence enters a region containing matter, for which Φ_{00} is non-zero. Since the energy is non-negative, $\Phi_{00} \geq 0$, and equation (5.1*a*) implies that ρ must become increasingly positive. This indicates that the congruence must start to contract. Eventually this congruence will focus. These properties have been described in detail by Penrose (1966). A congruence passing through a region of non-zero energy density is focused by it. It is even possible to measure the magnitude of the energy density by its focusing power.

Consider also an alternative situation in which the congruence enters a region with non-zero Ψ_0. Effectively this means that the congruence meets an opposing gravitational wave. In this case equation (5.1*b*) implies that the congruence starts to shear, the shear axis being determined by the polarization of the gravitational wave. This introduces the non-negative

term $\sigma\bar{\sigma}$ into equation (5.1a), and this in turn induces the congruence to contract. This process has been interpreted by Penrose (1966) in terms of the astigmatic focusing of the congruence by the gravitational wave. He has suggested that the amplitude and polarization of a gravitational wave could also be measured by its focusing properties.

These properties can easily be demonstrated using the plane wave solutions described in the previous chapter, though it is convenient here to define the wave fronts by v =constant, rather than u =constant. This enables us to consider a null congruence which extends into the wave, and on which u, x and y are constant.

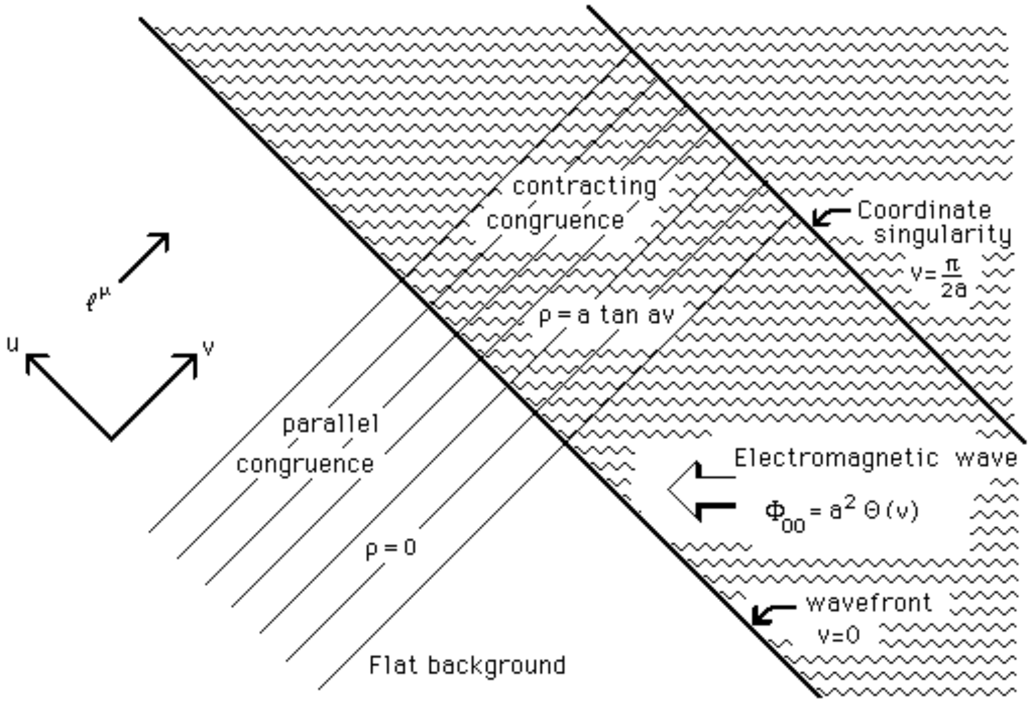


Figure 5.1 An initially parallel congruence meets an opposing electromagnetic wave and is focused by it. The wave front is taken to be the null hypersurface $v = 0$, and the contraction of the opposing congruence becomes unbounded on the hypersurface $v = \pi/2a$. The focusing occurs in the x–y-plane which is not indicated in the picture.

Consider first a parallel congruence in a flat region of space-time which extends into an opposing electromagnetic shock wave as illustrated in Figure 5.1. It is convenient to choose the tangent vector of this congruence to be l^μ. The wave front of the electromagnetic field may be taken to be $v = 0$, so that the energy-momentum tensor of the field is given by $\Phi_{00} = a^2\Theta(v)$. This describes an electromagnetic step wave with a propagation vector n^μ. The complete space-time can be described by the

line element (4.18), but with v replacing u. Transforming this to Rosen form, the flat region in which $v < 0$ has the line element (4.19), and the region $v \geq 0$ which contains the electromagnetic wave is described by

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 av(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{5.2}$$

The tetrad may be taken to be

$$l_\mu = u_{,\mu}, \quad n_\mu = v_{,\mu}, \quad m_\mu = \tfrac{1}{\sqrt{2}}\cos av(x_{,\mu} + iy_{,\mu}) \tag{5.3}$$

and the only non-zero spin coefficient is

$$\rho = a\tan av. \tag{5.4}$$

This indicates that the congruence tangent to l^μ increasingly contracts after passing the electromagnetic wave front. This clearly illustrates a case of pure focusing. Notice that the contraction becomes unbounded on the hypersurface $av = \pi/2$ on which the metric is singular. The focal plane thus coincides with the coordinate singularity of the metric. This is illustrated in Figure 5.1.

Now consider the alternative situation in which the parallel null congruence meets an opposing gravitational shock wave as illustrated in Figure 5.2. Using a similar notation, the gravitational wave can be described by $\Psi_0 = a^2\Theta(v)$, having the wave front $v = 0$. The complete space-time is described by the line element (4.22), but again with v replacing u. In Rosen form, the flat region $v < 0$ has the line element (4.19), and the region $v \geq 0$ which contains the gravitational wave is now described by

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 av\,\mathrm{d}x^2 - \cosh^2 av\,\mathrm{d}y^2. \tag{5.5}$$

Using the tetrad

$$l_\mu = u_{,\mu}, \quad n_\mu = v_{,\mu}, \quad m_\mu = \tfrac{1}{\sqrt{2}}(\cos av x_{,\mu} + i\cosh av y_{,\mu}) \tag{5.6}$$

the non-zero spin coefficients are

$$\begin{aligned} \rho &= \tfrac{1}{2}a(\tan av - \tanh av) \\ \sigma &= \tfrac{1}{2}a(\tan av + \tanh av). \end{aligned} \tag{5.7}$$

In this case the congruence tangent to l^μ, on which u, x and y are constants, both contracts and shears after passing the gravitational wave front. This illustrates a case of astigmatic focusing, the shear axes being aligned here with the x and y axes. Notice that both the contraction and

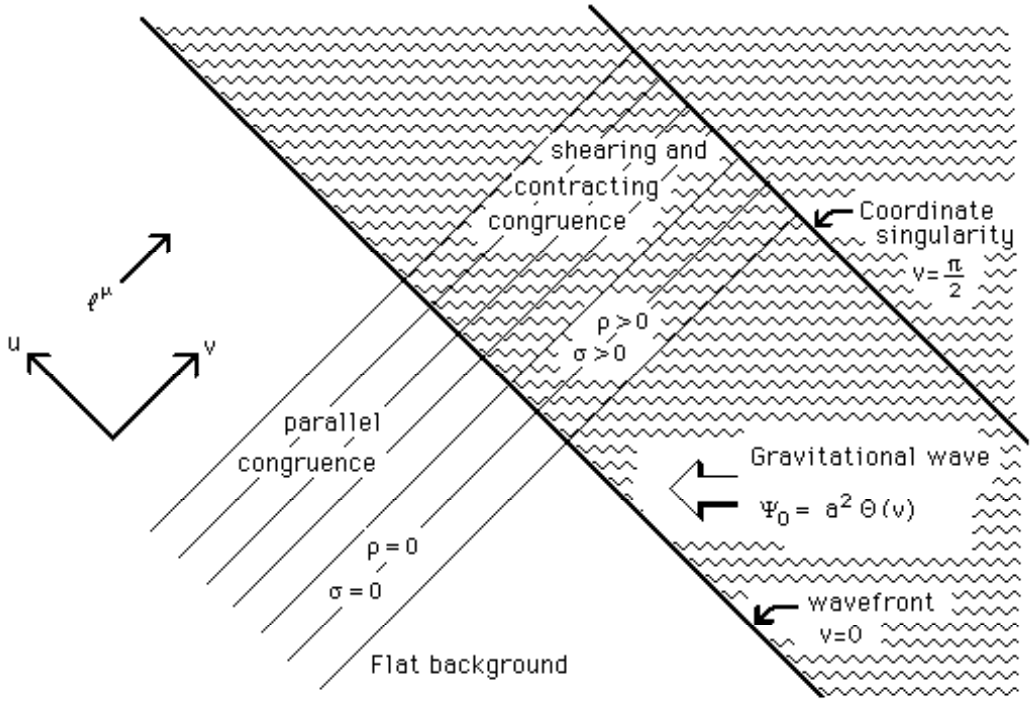


Figure 5.2 An initially parallel congruence meets an opposing gravitational wave and is focused astigmatically. The wave front is given by $v = 0$, and the contraction and shear of the opposing congruence become unbounded as $v \to \pi/2a$. The focusing occurs in the x-direction which, together with the y-direction is not indicated in the picture.

the shear become unbounded on the hypersurface $av = \pi/2$, as indicated in Figure 5.2.

It may also be noticed that, since the polarization vector is here aligned with the x-coordinate direction, this is the direction in which the contraction occurs, producing the singularity in the line element (5.5).

A very similar situation occurs when a parallel congruence meets an impulsive gravitational wave. The fact that the space-time is again flat behind the wave does not alter its focusing properties. The line element equivalent to (4.25) is still singular on the hypersurface on which the opposing congruence is focused.

These general results indicate that parallel congruences that extend into plane waves will tend to produce caustics in the region behind the wave fronts. These caustics, which arise from the focusing properties of the congruences, will normally be associated with coordinate singularities.

5.2 General theorems

There are a number of well known theorems in general relativity that have a bearing on the geometry of congruences and the associated components

of the curvature tensor. It is appropriate to state some of these here as they enable us to build up an intuitive feeling for colliding wave problems.

The best known of these theorems is due to Goldberg and Sachs (1962). It can be stated as follows:

> **Theorem 5.1** (The Goldberg–Sachs theorem) *A source-free space-time is algebraically special if, and only if, it possesses a shear-free geodesic null congruence.*

It is possible to align the vector field l^μ either with the repeated null direction of the gravitational field, or with the shear-free geodesic null congruence. The Goldberg–Sachs theorem then essentially states that, if $\Phi_{AB} = 0$, $\Lambda = 0$ then

$$\Psi_0 = \Psi_1 = 0 \quad \Leftrightarrow \quad \sigma = \kappa = 0. \tag{5.8}$$

This can easily be proved using Newman–Penrose techniques. Generalizations of this theorem with fewer restrictions on the Ricci tensor have been obtained by Kundt and Trümper (1962), and Kundt and Thompson (1962).

The Goldberg–Sachs theorem can be interpreted as stating that single algebraically special gravitational waves propagate along shear-free geodesic null congruences. The congruences opposing the wave, however, necessarily shear. It has been shown in the above section that, if $\Psi_0 \neq 0$, then $\sigma \neq 0$. It follows that the different gravitational wave components cannot be simply superposed.

A related theorem applying to Einstein–Maxwell fields has been obtained by Mariot (1954) and Robinson (1961). This can be stated in the following way.

> **Theorem 5.2** (The Mariot–Robinson theorem) *The repeated principal null direction of a null electromagnetic field is necessarily tangent to a shear-free geodesic null congruence, and is also a repeated principal null direction of an algebraically special gravitational field.*

If l^μ is aligned with the repeated principal null direction of the electromagnetic field, this theorem can be sumarized by the statement

$$\Phi_0 = \Phi_1 = 0, \ \Phi_2 \neq 0 \quad \Rightarrow \quad \sigma = \kappa = 0, \ \Psi_0 = \Psi_1 = 0. \tag{5.9}$$

The first part of this theorem, which states that a null electromagnetic field necessarily propagates along a shear-free geodesic null congruence, follows immediately from Maxwell's equations in the form (2.18). The remainder can easily be obtained using Newman–Penrose techniques.

One further theorem is of relevance to the interacting wave problem. This was obtained by Szekeres (1965).

Theorem 5.3 (The Szekeres theorem) *No solutions of Einstein's source-free field equations exist for which* $\Psi_0 \neq 0$, $\Psi_4 \neq 0$, *with* $\Psi_1 = \Psi_2 = \Psi_3 = 0$.

This is easily proved using the Newman–Penrose formalism. It follows immediately from Theorem 5.3 that, in a vacuum space-time, two transverse gravitational waves cannot be simply superposed. For the type of interactions considered in this book, in which the approaching gravitational waves are described by the components Ψ_0 and Ψ_4, it is clear that other components of the gravitational field must be induced as part of the interaction between such waves. A similar theorem proving the non-superposition of two longitudinal gravitational waves has been obtained elsewhere (Griffiths 1975*a*).

5.3 Colliding waves

The theorems in the previous section enable us immediately to gain some insight into some of the general properties of colliding wave problems. The basic situation may be considered as described in Figure 3.1, in terms of a flat background in region I, two approaching waves in regions II and III, and an interaction region IV following the collision.

Consider first the collision of a pair of plane gravitational waves. Initially, prior to the collision, each wave is of type N, and propagates along shear-free geodesic null congruences according to the Goldberg–Sachs theorem 5.1. However, congruences in the opposite direction to each wave necessarily shear. Thus, as the two waves meet and begin to pass through each other, they start to shear and contract. They begin to focus each other astigmatically.

It is possible to align the two vectors l^μ and n^μ with the propagation vectors of the approaching gravitational waves. The waves are thus described initially by the components Ψ_4 and Ψ_0 respectively. It is then clear from the Szekeres theorem 5.3, that the interaction region cannot be characterized solely by these two components, additional terms must appear. We have already seen in the Khan–Penrose solution (3.10), that it is the Ψ_2 term which appears. In fact, as will be shown later, this is the usual case.

It may also be worth pointing out that we would expect the gravitational field in the interaction region to be algebraically general. Clearly the two vectors l^μ and n^μ are not aligned with principal null directions of the Weyl tensor in the interaction region, though we may still think

of them as being aligned with the two initial wave components. There are, however, some exceptions and degenerate cases that are algebraically special will be considered later.

Now consider the collision of a pair of electromagnetic waves. In regions II and III, each will propagate along null geodesics having expansion twist and shear all zero. Null congruences passing through them in opposite directions are focused anastigmatically. Thus, as the two waves meet, we would initially expect them to focus each other without the introduction of shear. This, however, is not the case. In fact, it will be shown later that shear terms necessarily appear, and so the two waves must focus each other astigmatically. The introduction of shear, however, consequently introduces non-zero components of the Weyl tensor. In this way, the collision of electromagnetic waves can be considered to generate gravitational waves.

Finally, consider the collision of a gravitational wave with an electromagnetic wave. Let region II contain the initial gravitational wave with component Ψ_4, and let region III contain the approaching electromagnetic wave with component Φ_0. Congruences on which $u =$ constant are focused in region III, where $\rho \neq 0$, but $\sigma = 0$. We would therefore expect the gravitational wave to be focused anastigmatically as it enters the electromagnetic wave. In contrast, congruences on which $v =$ constant are focused astigmatically in region II. The non-zero contraction and shear are here given by $-\mu$ and $-\lambda$. We would therefore expect the electromagnetic wave to be focused astigmatically as it enters the gravitational wave. However, Maxwell's equations (2.18) would then become inconsistent if Φ_0 were the only non-zero component of the electromagnetic field. It therefore follows that other components of the electromagnetic field must appear in the interaction region.

This result is not unexpected. It follows from the Mariot–Robinson theorem 4.2 that a null electromagnetic wave must propagate along a shear-free geodesic null congruence. As the electromagnetic wave meets the gravitational wave, this congruence must start to shear, and therefore the electromagnetic field cannot remain null. The electromagnetic wave must therefore be partially reflected by the gravitational wave. This feature of the backscattering of the electromagnetic wave has been described by Penrose (1972).

We have now concluded that an electromagnetic wave must be partially reflected by a gravitational wave. But what about the gravitational wave: is that also partially reflected? To answer this question we will need to obtain exact solutions. In fact, the gravitational wave is not necessarily reflected. Exact solutions exist in which the Ψ_0 term is not generated in the interaction region, though the Ψ_2 term always appears. In these cases

the Weyl tensor in the interaction region is thus algebraically special. This simplifies the field equations significantly, and enables us to find a fairly general class of exact solutions. Given an arbitrary plane gravitational wave in region II, and a plane electromagnetic wave in region III, an exact solution in region IV can be obtained explicitly.

6

THE FIELD EQUATIONS

The colliding wave problem has been described in terms of two approaching waves in regions II and III, in a background region I, which is here taken to be flat. According to the work of Penrose (1980), the initial data are well set, so that a unique solution exists in the interaction region IV at least in the neighbourhood of the boundaries of regions II and III. It is therefore necessary first to state the relevant field equations in the interaction region, and then to attempt to solve them subject to the appropriate boundary conditions. This formal approach is not always achievable in practice, as analytic solutions in region IV are not easily obtained. It is often more appropriate to start with the field equations for region IV and, once a solution is found, subsequently to consider the particular initial waves that would give rise to it. In either case, it is necessary to formulate the appropriate field equations for the interaction region. The purpose of this chapter is to derive such equations.

Following Szekeres (1972) and Griffiths (1976*b*), the field equations will be derived here using Newman–Penrose techniques.

6.1 The coordinate system

The approaching waves have plane symmetry. On collision they will warp and scatter each other. In the interaction region they will no longer retain the full plane symmetry described by (4.15), though it may be expected that they will retain a two-parameter symmetry group of motions generated by the Killing vectors

$$\xi_1 = \partial_x, \qquad \xi_2 = \partial_y. \tag{6.1}$$

We expect the interaction region to retain plane symmetry in this sense. We therefore initially assume that the Killing vectors ξ_1 and ξ_2 exist throughout the space-time, and that the metric therefore has no explicit dependence on the coordinates x and y in the interaction region. Ultimately of course, this assumption is only justified if it leads to exact solutions.

At each point of the manifold there exist just two null directions orthogonal to the planes spanned by ∂_x and ∂_y. We may now align the two vectors l^μ and n^μ with these two directions. Since the congruences

in these directions are hypersurface orthogonal, the vectors l^μ and n^μ are proportional to gradients. It is therefore possible to put

$$l_\mu = A^{-1}u_{,\mu}, \qquad n_\mu = B^{-1}v_{,\mu} \tag{6.2}$$

where A and B are independent of x and y.

We may now choose u and v as suitable null coordinates in the interaction region. At this point, these are arbitrary functions of the coordinates with the same labels in regions II and III continued into the interaction region. It is therefore appropriate to assume that $u \geq 0$ and $v \geq 0$ in region IV. Labelling the coordinates as $(u,\ v,\ x,\ y) = (x^0,\ x^1,\ x^2,\ x^3)$, the null tetrad may now be chosen to be

$$\begin{aligned} l_\mu &= A^{-1}\delta^0_\mu, & l^\mu &= (0, B, Y^2, Y^3) \\ n_\mu &= B^{-1}\delta^1_\mu, & n^\mu &= (A, 0, X^2, X^3) \\ & & m^\mu &= (0, 0, \xi^2, \xi^3) \end{aligned} \tag{6.3}$$

where, putting $i = 2, 3$,

$$X^i = X^i(u,v), \qquad Y^i = Y^i(u,v), \qquad \xi^i = \xi^i(u,v). \tag{6.4}$$

The tetrad and coordinates are now defined up to the following transformations:
(1) Scale transformations or *boosts:*

$$l^{\mu\prime} = \phi l^\mu, \qquad n^{\mu\prime} = \phi^{-1}n^\mu, \qquad \phi = \phi(u,v), \tag{6.5}$$

under which the scale functions A and B transform as

$$A' = \phi^{-1}A, \qquad B' = \phi B.$$

(2) Spatial rotations or *spins:*

$$m^{\mu\prime} = e^{iC}m^\mu, \qquad C = C(u,v). \tag{6.6}$$

This is a rotation in a space-like 2-surface and transforms the ξ^i by

$$\xi^{i\prime} = e^{iC}\xi^i.$$

(3) Relabelling of null hypersurfaces:

$$u' = f(u), \qquad v' = g(v), \tag{6.7}$$

which induces the transformation in the scale functions

$$A' = f_{,u}A\ , \qquad B' = g_{,v}B.$$

(4) Spatial coordinate transformations:

$$u' = u, \qquad v' = v, \qquad x^{i\prime} = x^i + h^i(u,v), \tag{6.8}$$

which transforms the X^i and Y^i components by

$$X^{i\prime} = X^i + h^i_{,u}A\ , \qquad Y^{i\prime} = Y^i + h^i_{,v}B.$$

(5) Linear coordinate transformations:

$$x^{i\prime} = a^i{}_j x^j, \qquad a^i{}_j = \text{consts.} \tag{6.9}$$

6.2 The derivation of the field equations

With the tetrad defined as above, it may be noticed that all components are at most functions of u and v only. It follows that the differential operators, when applied to spin coefficients, take the form

$$D = B\tfrac{\partial}{\partial v}, \qquad \Delta = A\tfrac{\partial}{\partial u}, \qquad \delta = 0. \tag{6.10}$$

The incoming waves have no dependence on the coordinates x and y, and so no such dependence is expected in the interaction region.

The metric equations, which are the commutation relations (2.15) applied to the coordinates, immediately take the form

$$\begin{aligned}
\kappa = \nu = 0, \qquad \rho = \bar\rho, \qquad \mu = \bar\mu& \\
\bar\tau = \pi = 2\bar\beta &= 2\alpha \\
DA &= -(\epsilon + \bar\epsilon)A \\
\Delta B &= (\gamma + \bar\gamma)B \\
\Delta Y^i - DX^i &= (\gamma + \bar\gamma)Y^i + (\epsilon + \bar\epsilon)X^i - 4\alpha\xi^i - 4\bar\alpha\bar\xi^i \\
D\xi^i &= (\rho + \epsilon - \bar\epsilon)\xi^i + \sigma\bar\xi^i \\
\Delta\xi^i &= -(\mu + \bar\gamma - \gamma)\xi^i - \bar\lambda\bar\xi^i.
\end{aligned} \tag{6.11}$$

The first of these equations simply expresses the condition, consistent with (6.2), that the congruences tangent to l^μ and n^μ are geodesic and

twist-free. The third and fourth metric equations (6.11) enable the real parts of ϵ and γ to be defined by

$$\epsilon+\bar{\epsilon}=-B(\log A)_{,v}\ ,\qquad \gamma+\bar{\gamma}=A(\log B)_{,u}. \tag{6.12}$$

It can thus be seen that the two null congruences do not necessarily have affine parametrization.

Before proceeding to the field equations, it is convenient to introduce modified 'scale-invariant' spin coefficients

$$\begin{array}{lll}
\rho^\circ=\rho B^{-1}, & \sigma^\circ=\sigma B^{-1}, & E^\circ=i(\bar{\epsilon}-\epsilon)B^{-1},\\
\mu^\circ=\mu A^{-1}, & \lambda^\circ=\lambda A^{-1}, & G^\circ=\ i(\bar{\gamma}-\gamma)A^{-1},\\
\alpha^\circ=\alpha(AB)^{-1/2}, & Y^{\circ i}=Y^iB^{-1}, & X^{\circ i}=\ X^iA^{-1},\\
\Psi_0^\circ=\Psi_0B^{-2}, & \Psi_1^\circ=\ \Psi_1A^{-1/2}B^{-3/2}, & \Psi_2^\circ=\Psi_2(AB)^{-1},\\
\Psi_3^\circ=\Psi_3B^{-1/2}A^{-3/2}, & \Psi_4^\circ=\ \Psi_4A^{-2}, & \Lambda^\circ=\Lambda(AB)^{-1},\\
\Phi_{00}^\circ=\Phi_{00}B^{-2}, & \Phi_{01}^\circ=\ \Phi_{01}A^{-1/2}B^{-3/2}, & \Phi_{02}^\circ=\Phi_{02}(AB)^{-1},\\
\Phi_{11}^\circ=\Phi_{11}(AB)^{-1}, & \Phi_{12}^\circ=\ \Phi_{12}B^{-1/2}A^{-3/2}, & \Phi_{22}^\circ=\Phi_{22}A^{-2}.
\end{array} \tag{6.13}$$

It is also convenient to introduce the scale-invariant function M, defined by

$$M=\log AB. \tag{6.14}$$

The metric equations (6.11) and the Ricci identities (2.14) now only involve quantities that are invariant with respect to the scale transformations (6.5). Together they give the following:

$$Y^{\circ i}{}_{,u}-X^{\circ i}{}_{,v}=-4(\alpha^\circ\xi^i+\bar{\alpha}^\circ\bar{\xi}^i)e^{-M/2} \tag{6.15a}$$
$$\xi^i{}_{,v}=(\rho^\circ+iE^\circ)\xi^i+\sigma^\circ\bar{\xi}^i \tag{6.15b}$$
$$\xi^i{}_{,u}=-(\mu^\circ-iG^\circ)\xi^i-\bar{\lambda}\bar{\xi}^i \tag{6.15c}$$
$$\rho^\circ{}_{,v}=\rho^{\circ 2}+\sigma^\circ\bar{\sigma}^\circ-\rho^\circ M_{,v}+\Phi_{00}^\circ \tag{6.15d}$$
$$\rho^\circ{}_{,u}=-2\rho^\circ\mu^\circ-4\alpha^\circ\bar{\alpha}^\circ-\Phi_{11}^\circ-3\Lambda^\circ \tag{6.15e}$$
$$\mu^\circ{}_{,v}=2\rho^\circ\mu^\circ+4\alpha^\circ\bar{\alpha}^\circ+\Phi_{11}^\circ+3\Lambda^\circ \tag{6.15f}$$
$$\mu^\circ{}_{,u}=-\mu^{\circ 2}-\lambda^\circ\bar{\lambda}^\circ-\mu^\circ M_{,u}-\Phi_{22}^\circ \tag{6.15g}$$
$$\sigma^\circ{}_{,v}=\sigma^\circ(2\rho^\circ-M^\circ_{,v}+2iE^\circ)+\Psi_0^\circ \tag{6.15h}$$
$$\sigma^\circ{}_{,u}=-\sigma^\circ(\mu^\circ-2iG^\circ)-\rho^\circ\bar{\lambda}^\circ-4\bar{\alpha}^{\circ 2}-\Phi_{02}^\circ \tag{6.15i}$$
$$\lambda^\circ{}_{,v}=\lambda^\circ(\rho^\circ-2iE^\circ)+\mu^\circ\bar{\sigma}^\circ+4\alpha^{\circ 2}-\Phi_{20}^\circ \tag{6.15j}$$
$$\lambda^\circ{}_{,u}=-\lambda^\circ(2\mu^\circ+M_{,u}+2iG^\circ)-\Psi_4^\circ \tag{6.15k}$$

$$\alpha^\circ{}_{,v} = \alpha^\circ(3\rho^\circ - \tfrac{1}{2}M_{,v} - iE^\circ) + \bar\alpha^\circ\bar\sigma^\circ + \Phi^\circ_{10} \tag{6.15l}$$

$$\alpha^\circ{}_{,u} = -\alpha^\circ(3\mu^\circ + \tfrac{1}{2}M_{,u} + iG^\circ) - \bar\alpha^\circ\lambda^\circ - \Phi^\circ_{21} \tag{6.15m}$$

$$\tfrac{1}{2}M_{,uv} + \tfrac{1}{2}i(G^\circ{}_{,v} - E^\circ{}_{,u}) = \rho^\circ\mu^\circ - \sigma^\circ\lambda^\circ + 12\alpha^\circ\bar\alpha^\circ + 2\Phi^\circ_{11} \tag{6.15n}$$

$$\Psi^\circ_1 = 2\rho^\circ\bar\alpha^\circ - 2\sigma^\circ\alpha^\circ + \Phi^\circ_{01} \tag{6.15o}$$

$$\Psi^\circ_2 = \rho^\circ\mu^\circ - \sigma^\circ\lambda^\circ + \Phi^\circ_{11} + \Lambda^\circ \tag{6.15p}$$

$$\Psi^\circ_3 = 2\mu^\circ\alpha^\circ - 2\lambda^\circ\bar\alpha^\circ + \Phi^\circ_{21} \tag{6.15q}$$

Following the method of Szekeres, it is convenient at this point to put

$$\begin{aligned} \xi^2 &= \tfrac{1}{\sqrt{2}}e^{(U-V)/2}(\cosh\tfrac{1}{2}W + i\sinh\tfrac{1}{2}W) \\ \xi^3 &= \tfrac{1}{\sqrt{2}}e^{(U+V)/2}(\sinh\tfrac{1}{2}W + i\cosh\tfrac{1}{2}W) \end{aligned} \tag{6.16}$$

where U, V and W are functions of u and v. Using equations (6.15b,c) some of the scale-invariant spin coefficients can then be expressed explicitly as

$$\begin{aligned} \rho^\circ &= \tfrac{1}{2}U_v & \mu^\circ &= -\tfrac{1}{2}U_u \\ E^\circ &= -\tfrac{1}{2}V_v\sinh W & G^\circ &= -\tfrac{1}{2}V_u\sinh W \\ \sigma^\circ &= \tfrac{1}{2}iW_v - \tfrac{1}{2}V_v\cosh W & \lambda^\circ &= \tfrac{1}{2}iW_u + \tfrac{1}{2}V_u\cosh W \end{aligned} \tag{6.17}$$

where the suffices u and v refer to partial derivatives taken with respect to these variables. From now on, the commas on metric functions will be omitted and derivatives will be denoted only by suffices.

The method of procedure from this point depends heavily on the form of the Ricci tensor. For colliding gravitational and electromagnetic waves it will be shown that Φ_{01} and Φ_{21} are both zero. In this case it follows from (6.15l,m) that, since $\alpha^\circ = 0$ in the initial regions II and III, it is necessary that $\alpha^\circ = 0$ everywhere.

By contrast, for colliding neutrino fields (Griffiths 1976a) which will only be considered in Section 20.5, the components Φ_{01} and Φ_{21} are essentially non-zero and the method of integrating the field equations differs significantly. In this, and similar cases, it is clearly not possible to put $\alpha^\circ = 0$ everywhere. Apart from section 20.5, however, the methods for integrating the field equations in these cases will not be pursued.

In this book the emphasis is on gravitational and electromagnetic waves, so only the Einstein–Maxwell equations will be derived explicitly in this section. Einstein's vacuum equations are obtained simply by putting the electromagnetic field components equal to zero.

6.3 The Einstein and Einstein–Maxwell equations

In order to consider the collision of electromagnetic waves, it is convenient first to define the scale-invariant quantities

$$\Phi_0^\circ = \Phi_0 B^{-1}, \qquad \Phi_1^\circ = \Phi_1 (AB)^{-1/2}, \qquad \Phi_2^\circ = \Phi_2 A^{-1}. \tag{6.18}$$

With the conditions (6.11), Maxwell's equations (2.18) now become

$$\begin{aligned}
\Phi_{1,v}^\circ &= (2\rho^\circ - \tfrac{1}{2}M_{,v})\Phi_1^\circ \\
\Phi_{2,v}^\circ &= -\lambda^\circ\Phi_0^\circ + 4\alpha^\circ\Phi_1^\circ + (\rho^\circ - iE^\circ)\Phi_2^\circ \\
\Phi_{0,u}^\circ &= -(\mu^\circ - iG^\circ)\Phi_0^\circ - 4\bar{\alpha}^\circ\Phi_1^\circ + \sigma^\circ\Phi_2^\circ \\
\Phi_{1,u}^\circ &= -(2\mu^\circ - \tfrac{1}{2}M_{,u})\Phi_1^\circ .
\end{aligned} \tag{6.19}$$

It is immediately clear from the first and last of these equations that, since Φ_1 is zero in regions II and III, it must also be zero in the interaction region. It follows that Φ_{01}° and Φ_{21}° are both zero for colliding electromagnetic waves, as they obviously are for pure gravitational waves. With this condition, equations (6.15 l, m) imply that α° is also zero in region IV, since it is similarly zero in regions II and III.

Equation (6.15a) now implies that, since $\alpha = 0$, it is possible to use a spatial coordinate transformation (6.8) to make both X^i and Y^i zero simultaneously. With this transformation, the metric now takes the form

$$\begin{aligned}
\mathrm{d}s^2 = {} & 2e^{-M}\mathrm{d}u\mathrm{d}v \\
& - e^{-U}(e^V \cosh W \mathrm{d}x^2 - 2\sinh W \mathrm{d}x\mathrm{d}y + e^{-V}\cosh W \mathrm{d}y^2).
\end{aligned} \tag{6.20}$$

This will be referred to as the Szekeres line element.

The only non-zero spin coefficients are now ρ, σ, ϵ, μ, λ and γ. Apart from the freedom associated with the scale transformation (6.5), these are now expressed explicitly in terms of the metric functions by equations (6.17) and (6.12).

Using (6.17) the remaining components of Maxwell's equations (6.19) now become

$$\begin{aligned}
\Phi_{2,v}^\circ &= \tfrac{1}{2}(U_v + iV_v \sinh W)\Phi_2^\circ - \tfrac{1}{2}(iW_u + V_u\cosh W)\Phi_0^\circ \\
\Phi_{0,u}^\circ &= \tfrac{1}{2}(U_u - iV_u \sinh W)\Phi_0^\circ + \tfrac{1}{2}(iW_v - V_v\cosh W)\Phi_2^\circ .
\end{aligned} \tag{6.21}$$

The expressions (6.17) may also be substituted back into equations (6.15 d, e, f, g, i, j, n) to give

$$U_{uv} = U_u U_v \tag{6.22a}$$

$$\begin{aligned}
2U_{vv} &= {U_v}^2 + {W_v}^2 + {V_v}^2\cosh^2 W - 2U_vM_v + 4\Phi_0^\circ\bar\Phi_0^\circ &\quad (6.22b)\\
2U_{uu} &= {U_u}^2 + {W_u}^2 + {V_u}^2\cosh^2 W - 2U_uM_u + 4\Phi_2^\circ\bar\Phi_2^\circ &\quad (6.22c)\\
2V_{uv} &= U_uV_v + U_vV_u - 2(V_uW_v + V_vW_u)\tanh W \\
&\quad + 2(\Phi_0^\circ\bar\Phi_2^\circ + \Phi_2^\circ\bar\Phi_0^\circ)\mathrm{sech} W &\quad (6.22d)\\
2W_{uv} &= U_uW_v + U_vW_u + 2V_uV_v\sinh W\cosh W \\
&\quad + 2i(\Phi_0^\circ\bar\Phi_2^\circ - \Phi_2^\circ\bar\Phi_0^\circ) &\quad (6.22e)\\
2M_{uv} &= -U_uU_v + W_uW_v + V_uV_v\cosh^2 W. &\quad (6.22f)
\end{aligned}$$

These are Einstein's field equations written in component form.

The remaining equations (6.15 h, o, p, q, k) give the scale-invariant components of the Weyl tensor in the form

$$\begin{aligned}
\Psi_0^\circ &= -\tfrac{1}{2}((V_{vv} - U_vV_v + M_vV_v)\cosh W + 2V_vW_v\sinh W)\\
&\quad + \tfrac{1}{2}i(W_{vv} - U_vW_v + M_vW_v - {V_v}^2\cosh W\sinh W)\\
\Psi_1^\circ &= 0\\
\Psi_2^\circ &= \tfrac{1}{2}M_{uv} - \tfrac{1}{4}i(V_uW_v - V_vW_u)\cosh W \qquad (6.23)\\
\Psi_3^\circ &= 0\\
\Psi_4^\circ &= -\tfrac{1}{2}((V_{uu} - U_uV_u + M_uV_u)\cosh W + 2V_uW_u\sinh W)\\
&\quad - \tfrac{1}{2}i(W_{uu} - U_uW_u + M_uW_u - {V_u}^2\cosh W\sinh W).
\end{aligned}$$

It follows from this that it is only the Coulomb component Ψ_2 that appears as an interaction term when the gravitational waves Ψ_0 and Ψ_4 interact.

6.4 Integrating the field equations

Having derived field equations for region IV of the colliding plane wave situation, it is initially necessary to question the possible existence of solutions beyond the initial boundaries with regions II and III. This question has been considered by Yurtsever (1989*a*). In an appendix to this paper, he has given a formal proof of the global existence and uniqueness of solutions in the interaction region as far as the focusing singularity, subject to the boundary conditions which will be discussed in the next chapter.

With this question settled, it is appropriate here to make a few simple observations about the basic approach to the integration of the above field equations.

To start with, it can be seen that equation (6.22a) can immediately be integrated to give

$$e^{-U} = f(u) + g(v) \qquad (6.24)$$

where $f(u)$ and $g(v)$ are arbitrary functions that, at this stage, may be assumed to be piecewise C^1.

It may also initially be observed that the Szekeres line element (6.20) is similar to that for the plane wave metric in Rosen form (4.13) that is appropriate for regions II and III, except that the metric functions are here functions of both u and v. In fact it is possible to use (6.20) to describe the space-time in all four regions, where U, V, W and M are functions of u and v in region IV, are functions of u only in region II, are functions of v only in region III, and are constants that can be transformed to be zero in region I. Regions II and III contain approaching plane waves that only differ from those described in Chapter 4 by a simple coordinate transformation (6.7) in u or v.

Using this approach, it is appropriate to choose f and g such that

$$\begin{aligned} f &= \tfrac{1}{2} \qquad \text{for} \qquad u \leq 0 \\ g &= \tfrac{1}{2} \qquad \text{for} \qquad v \leq 0. \end{aligned} \tag{6.25}$$

The approaching waves are thus partially described by the functions $f(u)$ in region II, and $g(v)$ in region III. Of the remaining functions, M can be removed in regions II and III by the transformation (6.7), and V and W must satisfy either (6.22c) or (6.22b). The approaching waves are thus characterized by two independent functions in each region, which together describe the amplitude and polarization of the two approaching waves.

Returning to consider the full field equations in region IV, it may be observed that equations (6.22d, e), together with Maxwell's equations (6.21), are the integrability conditions for the field equations (6.22 b, c, f). Thus if functions U, V, W, Φ_2° and Φ_0° are found to satisfy (6.22a, d, e) and (6.21), then a function M that satisfies the remaining equations (6.22b, c, f) automatically exists. It should not be assumed that this function should be zero. It is not easy to find analytic solutions of equations (6.22d, e) and (6.21) and, once a solution is found, a particular $M(u, v)$ can be obtained by integrating (6.22b, c). In view of the required continuity across the boundaries $u = 0$ and $v = 0$, non-zero functions $M(u)$ and $M(v)$ will usually occur in regions II and III respectively. Normally, it will be possible to reduce these to zero only in these regions, and then only for purposes of interpreting the solution obtained.

It may be seen from this discussion that, in order to integrate the field equations, attention is focused on equations (6.22 d, e) and (6.21) or, for colliding gravitational waves, just on (6.22 d, e). The emphasis is therefore on these equations as we seek to derive the exact solutions given in subsequent chapters.

7

BOUNDARY CONDITIONS

The colliding plane wave problem has been formulated in the previous chapters as a characteristic initial value problem with initial data specified on two null hypersurfaces. At this point it is necessary to discuss the junction conditions between the four regions described in Figure 3.1, and particularly on the initial boundaries of region IV.

7.1 General discussion

When pasting together particular solutions of Einstein's equations applying to different regions of space-time, it has generally been thought that the appropriate junction conditions are those of Lichnerowicz (1955). These require that there exist coordinates in which the metric tensor is C^1 and piecewise at least C^2. These conditions appear to be reasonable since the curvature tensor involves the second derivatives of the metric tensor.

Work on junction conditions up to 1966 has been surveyed by Israel (1966), with particular emphasis on conditions across non-null boundaries. He has also shown that, if the distributional part of the energy-momentum tensor is zero, then the second fundamental forms induced by the metric match across the boundary, and this implies that the full Riemann tensor is regular. This condition was proposed by Darmois (1927), and has been shown to be equivalent to the Lichnerowicz conditions by Bonnor and Vickers (1981).

An alternative set of conditions has been proposed by O'Brien and Synge (1952). In the non-null case, these also have been shown by Israel (1958) and Robson (1972) to be equivalent to those of Lichnerowicz. However, since these are not stated covariantly, some differences may occur as discussed by Bonnor and Vickers (1981).

O'Brien and Synge (1952) have also proposed conditions across null boundaries that are weaker than those of Lichnerowicz. Denoting the null hypersurface by $x^0 =$ constant, with $g_{00} = 0$, they propose simply that the components

$$g_{\mu\nu}, \qquad g^{ij}g_{ij,0}, \qquad g^{i0}g_{ij,0} \tag{7.1}$$

where $(i, j = 1, 2, 3)$, should be continuous across $x^0 =$ constant.

By requiring that the metric tensor be C^1 and piecewise at least C^2, the Lichnerowicz conditions specifically exclude the impulsive gravitational waves that are discussed in Chapter 3. It can be shown, however, that the line element (3.3) does satisfy the O'Brien–Synge conditions across the null hypersurface $u = 0$. This connects the metrics (3.6) and (3.7) in regions I and II. The importance of an appropriate choice of coordinates may also be pointed out here, since, when the continuous line element (3.3) is transformed to the form (3.1), the metric tensor then contains a delta function discontinuity.

It has been shown by Robson (1973), that the appropriate junction conditions across a null hypersurface are those of O'Brien and Synge (1952). Junction conditions across null hypersurfaces have been further analysed by Penrose (1972) and Clarke and Dray (1987). In addition, Bell and Szekeres (1974) have shown that, for colliding plane electromagnetic waves, the Lichnerowicz conditions have to be relaxed in favour of those of O'Brien and Synge.

7.2 Junction conditions for colliding plane waves

In order to discuss the collision of plane waves, it has been found convenient to divide space-time up into four regions as described in Figure 3.1. These regions are bounded by the two null hypersurfaces $u = 0$ and $v = 0$. It has also been suggested in Section 6.4, that it is convenient to assume that the line element (6.20) applies to the entire space-time. However, the metric functions U, V, W and M must take different forms in the four regions.

It is now necessary to consider the conditions that should be imposed on these functions at the boundaries of the four regions. The O'Brien–Synge conditions (7.1) imply that V, W and M are continuous and that U is smooth across these null boundaries. However, according to (6.24),

$$U = -\log(f(u) + g(v)). \tag{7.2}$$

It is therefore necessary that $f(u)$ and $g(v)$ are at least C^1.

In order for the line element (6.20) to describe a collision of plane waves, it must be assumed that U, V, W and M are functions of u and v in region IV, are functions of u only in region II, are functions of v only in region III, and are constants in region I. Concentrating initially on the metric function $e^{-U} = f + g$, it is appropriate to choose

$$\begin{aligned} f &= \tfrac{1}{2} \qquad \text{for} \ \ u \le 0, \qquad f'(0) = 0 \\ g &= \tfrac{1}{2} \qquad \text{for} \ \ v \le 0, \qquad g'(0) = 0. \end{aligned} \tag{7.3}$$

The approaching waves are then partly described by $f(u)$ in region II, and by $g(v)$ in region III. With this choice it may now be noted that

$$\begin{aligned} &\text{in region I :} && e^{-U}=1 \\ &\text{in region II :} && e^{-U}=\tfrac{1}{2}+f(u) \\ &\text{in region III :} && e^{-U}=\tfrac{1}{2}+g(v) \\ &\text{in region IV :} && e^{-U}=f(u)+g(v). \end{aligned} \tag{7.4}$$

The junction condition that U is smooth across the boundaries requires that the function $f(u)$ must have the same form in both regions II and IV. Similarly $g(v)$ must have the same form in regions III and IV.

At this point it is convenient to concentrate on the initial boundaries between regions I and II, and between regions I and III. It is then possible to use the transformations (6.7) to put $M=0$ in all three of these initial regions. Equations (6.22b,c), which now apply only to regions III and II respectively, become

$$\begin{aligned} g'' &= \tfrac{1}{2}e^{U}g'^2-\tfrac{1}{2}e^{-U}(V_v^2\cosh^2 W+W_v^2+4\Phi_0^\circ\bar\Phi_0^\circ) \\ f'' &= \tfrac{1}{2}e^{U}f'^2-\tfrac{1}{2}e^{-U}(V_u^2\cosh^2 W+W_u^2+4\Phi_2^\circ\bar\Phi_2^\circ). \end{aligned} \tag{7.5}$$

These equations, together with (7.3), imply that f and g are monotonically decreasing functions for positive arguments, at least in regions II and III. The scale transformations (6.7) cannot reverse the directions of the parameters u and v. Thus, even in region IV with M non-zero, f and g must be monotonically decreasing functions.

It is always possible, therefore, to use the transformation (6.7) to express the functions f and g in the forms

$$f=\tfrac{1}{2}-(c_1u)^{n_1}\Theta(u), \qquad g=\tfrac{1}{2}-(c_2v)^{n_2}\Theta(v) \tag{7.6}$$

which apply globally. It is also possible to use further transformations to put $c_1=1$ and $c_2=1$. However, these particular constants describe a measure of the magnitudes of the approaching waves, and it is therefore often convenient to retain them without this further rescaling. For situations in which $M=0$ in the inital regions I, II and III, the product c_1c_2 has an invariant meaning as the amount of non-linearity in the interaction. However, if transformations are used to put $c_1=1$ and $c_2=1$, then this magnitude is absorbed into the metric component e^{-M}.

The expressions (7.6) may be of particular convenience, as they transparently demonstrate the technique by which an exact solution obtained in region IV in terms of these functions can be extrapolated back to regions III, II and I. However, this particular parametrization is not always

the most convenient, as will be demonstrated for example in the Bell–Szekeres solution to be described in Chapter 15.

It may also be noted that in the line element (6.20) the metric coefficient e^{-U}, according to (6.24), is given by $f(u)+g(v)$ where f and g have now both been shown to be decreasing functions from the value $\frac{1}{2}$. It is therefore inevitable that a singularity will develop as $f+g \to 0$. Whether this is a curvature singularity or simply a coordinate singularity will have to be considered in detail. An initial discussion of this topic is given in the next chapter.

In practice, it is not easy to find exact solutions in the interaction region for any specified initial conditions in which the metric functions in regions I, II and III are the initial data. Although this approach will be discussed later, most of the explicit exact solutions that will be described in the following chapters have been obtained by first solving the field equations in region IV, and then extrapolating back to determine the approaching waves that would give rise to them. In this context it may be noted that, since f and g are monotonically decreasing functions, they may be adopted as coordinates in the interaction region. Also, since these functions are required to be smooth across the boundaries, they may easily be matched to the null coordinates u and v in regions II and III.

For colliding gravitational waves, attention is concentrated on equations $(6.22d, e)$. In the interaction region it is possible to use f and g as coordinates and these equations may then be integrated to give $V(f,g)$ and $W(f,g)$ subject to the initial data given by $V(\frac{1}{2},g)$ and $W(\frac{1}{2},g)$ on the surface $u=0$ and by $V(f,\frac{1}{2})$ and $W(f,\frac{1}{2})$ on the surface $v=0$. The situation for colliding electromagnetic waves is a little more complicated with the addition of (6.21) as well as the extra terms in $(6.22d, e)$. However, even in this case the junction conditions for V and W do not usually impose additional complications.

Once V and W are found from equations $(6.22d, e)$, it is then necessary to integrate $(6.22b, c)$ to obtain M. Because the integrability conditions are automatically satisfied, such a solution is known to exist.

Equations $(6.22b, c)$ can now be written in the form

$$
\begin{aligned}
M_v &= -\frac{g''}{g'} + \frac{g'}{2(f+g)} - \frac{(f+g)}{2g'}\left(V_v^2\cosh^2 W + W_v^2 + 4\Phi_0^\circ\bar{\Phi}_0^\circ\right) \\
M_u &= -\frac{f''}{f'} + \frac{f'}{2(f+g)} - \frac{(f+g)}{2f'}\left(V_u^2\cosh^2 W + W_u^2 + 4\Phi_2^\circ\bar{\Phi}_2^\circ\right).
\end{aligned}
\tag{7.7}
$$

It is thus convenient to put

$$
e^{-M} = \frac{f'g'}{\sqrt{f+g}}\, e^{-S} \tag{7.8}
$$

where S satisfies

$$\begin{aligned} S_g &= -\tfrac{1}{2}(f+g)\left(V_g^2\cosh^2 W + W_g^2 + 4\frac{\Phi_0^\circ\bar{\Phi}_0^\circ}{g'^2}\right) \\ S_f &= -\tfrac{1}{2}(f+g)\left(V_f^2\cosh^2 W + W_f^2 + 4\frac{\Phi_2^\circ\bar{\Phi}_2^\circ}{f'^2}\right). \end{aligned} \tag{7.9}$$

The boundary conditions discussed above require that e^{-M} be continuous, and that $f(0)=\frac{1}{2}$, $f'(0)=0$, $g(0)=\frac{1}{2}$ and $g'(0)=0$. However, in view of (7.8), e^{-M} cannot be continuous across $u=0$ and $v=0$ unless e^{-S} is unbounded on these boundaries. For the junction conditions to be satisfied, it is therefore essential that the functions V, W, Φ_0° and Φ_2°, which are solutions of equations (6.22d,e) and (6.21), should be such that the solution of (7.9) for S must contain terms of the form

$$S = k_1\log(\tfrac{1}{2}-f) + k_2\log(\tfrac{1}{2}-g) + \log(p_1p_2) + \dots \tag{7.10}$$

where p_1 and p_2 are constants. If the leading terms in a power series expansion for f and g in region IV have the form

$$f = \tfrac{1}{2} - (c_1u)^{n_1} + \dots, \qquad g = \tfrac{1}{2} - (c_2v)^{n_2} + \dots \tag{7.11}$$

then e^{-M} is continuous across the boundaries only if S contains the terms in (7.10) where the constants k_1 and k_2 are given by

$$k_1 = 1 - 1/n_1, \qquad k_2 = 1 - 1/n_2. \tag{7.12}$$

In order to satisfy (7.3), it is necessary that $n_1 \geq 2$ and $n_2 \geq 2$. It therefore follows that k_1 and k_2 must be restricted to the range

$$\tfrac{1}{2} \leq k_1, k_2 < 1. \tag{7.13}$$

In addition, in order to put M zero in region I, it is also appropriate to put

$$p_1 = -n_1c_1, \qquad p_2 = -n_2c_2. \tag{7.14}$$

It can be seen that the boundary conditions that are appropriate for colliding plane waves become fairly complicated. It is mainly a question of choosing solutions V, W, Φ_0° and Φ_2° of equations (6.22d,e) and (6.21) such that the solution M of (6.22b,c) contains terms of the form (7.10). Even then, the restrictions (7.12) are placed on the parameters and hence on the forms of f and g.

Since the main field equations involve the functions V, W, Φ_0° and Φ_2°, it is convenient to consider a form of the boundary conditions that applies to these functions only. This can be achieved by substituting (7.10) into (7.9) and considering boundaries as $f \to 1/2$ and $g \to 1/2$. The resulting conditions can be conveniently expressed in the form

$$\begin{aligned} \lim_{g\to 1/2}\left[(\tfrac{1}{2}-g)\left(V_g^2\cosh^2 W + W_g^2 + 4\frac{\Phi_0^\circ\bar{\Phi}_0^\circ}{g'^2}\right)\right] &= 2k_2 \\ \lim_{f\to 1/2}\left[(\tfrac{1}{2}-f)\left(V_f^2\cosh^2 W + W_f^2 + 4\frac{\Phi_2^\circ\bar{\Phi}_2^\circ}{f'^2}\right)\right] &= 2k_1 \end{aligned} \tag{7.15}$$

where k_1 and k_2 must satisfy (7.13). It is sometimes more convenient to revert to expressions involving the null coordinates u and v. These become

$$\begin{aligned} \lim_{v\to 0}\left[\frac{V_v^2\cosh^2 W + W_v^2 + 4\Phi_0^\circ\bar{\Phi}_0^\circ}{v^{n_2-2}}\right] &= 2n_2(n_2-1)c_2^{n_2} \\ \lim_{u\to 0}\left[\frac{V_u^2\cosh^2 W + W_u^2 + 4\Phi_2^\circ\bar{\Phi}_2^\circ}{u^{n_1-2}}\right] &= 2n_1(n_1-1)c_1^{n_1}. \end{aligned} \tag{7.16}$$

This form of the conditions is particularly convenient when $n_1 = n_2 = 2$.

These conditions will be discussed further for colliding gravitational waves in Section 11.2 and for the collision of a mixture of electromagnetic and gravitational waves in Section 16.1.

Before concluding this chapter, it is appropriate to consider the possibilities that arise when the continuity conditions for U are slightly relaxed, requiring only that it is continuous so that $n_1, n_2 \geq 1$. In this case, impulsive components occur in the Ricci tensor on the boundaries between the different regions. It is then possible that such components could be interpreted in terms of impulsive null matter fields. Explicit solutions of this type have in fact been given by Dray and 't Hooft (1986) and Tsoubelis (1989) and will be discussed in Section 20.4. However, the physical interpretation of such situations requires careful analysis. At this point, it may simply be pointed out that, in order for the matter tensor to have positive energy density, f' and g' must be negative on the boundaries.

8

SINGULARITY STRUCTURE

It has been seen that singularities inevitably occur in the solutions describing the interaction region of colliding plane waves. Using the line element (6.20), we have in this region

$$e^{-U} = f(u) + g(v) \tag{8.1}$$

where $f(u)$ and $g(v)$ are monotonically decreasing functions for positive arguments. It is therefore inevitable that some kind of singularity will occur on the hypersurface $f + g = 0$. It can also be seen from (6.17), that this is the hypersurface on which the two opposing waves mutually focus each other, as the contraction of each wave here becomes unbounded. This was anticipated by the discussion in Section 5.3. It is now appropriate to consider whether the caustics formed in this way correspond to mere coordinate singularities, or whether they are necessarily curvature singularities as they are in the Khan–Penrose solution.

It has also been pointed out previously that coordinate singularities necessarily occur in the regions II and III that contain the approaching waves. The character and significance of these singularities must also be considered in this chapter.

8.1 Singularities

According to the general theory of relativity, space-time is represented by a connected C^∞ Hausdorff manifold M together with a locally Lorentz metric g. A singularity[1] in the space-time is indicated by incomplete geodesics or incomplete curves of bounded acceleration (Hawking and Ellis, 1973).

By definition, space-time is smooth and does not contain any irregular points. It follows that a singularity may normally be considered as occurring only at a boundary of space-time. Unfortunately, no single definition of a singularity has yet been found which is applicable to all situations.

One of the more useful ways of attaching a boundary to a singular space-time is by a b(bundle)-boundary construction. The b-boundary

[1] This brief review follows closely that of Konkowski and Helliwell, 1989. For a more complete review see Tipler, Clarke and Ellis (1980)

is the projection into a space-time of a natural boundary attached to a higher-dimensional Riemannian manifold. In the standard b-boundary construction, the Riemannian manifold is the bundle of frames over space-time having a positive definite metric induced by the affine connection. Boundary points of the frame bundle are determined by giving end points to all Cauchy sequences which do not converge in the frame bundle. The bundle boundary is then projected down to make a boundary for the space-time.

According to the classification scheme devised by Ellis and Schmidt (1977), singularities in maximal, four-dimensional space-times can be divided into three basic types: quasiregular, non-scalar curvature and scalar curvature. This scheme describes the singularity structure of a space-time (M, g) on which the Riemann tensor is C^k. It uses a b-boundary construction to determine the location of singular points.

If the b-boundary is non-empty, there are only two possibilities. Either a point q in the b-boundary may be a C^r $(r \geq 0^-)$ regular boundary point if the space-time (M, g) can be embedded in a larger space-time (M', g') such that the Riemann tensor is C^r and q is an interior point in M', or it may be a C^r singular boundary point if the space-time (M, g) is not extendable through q in a C^r way.

A singular boundary point q can then be classified according to this scheme. It may be a C^k (or C^{k-}) *quasiregular singularity* if all components of the Riemann tensor and its first k derivatives evaluated in an orthonormal frame parallel propagated along an incomplete geodesic ending at q are C^0 (or C^{0-}). Such frames are called PPON frames. It may alternatively be a C^k (or C^{k-}) curvature singularity if this is not true. In this case it may either be categorized as a C^k (or C^{k-}) *non-scalar curvature singularity* if all scalars in the metric tensor $g_{\mu\nu}$, the alternating symbol $\epsilon_{\kappa\lambda\mu\nu}$, the Riemann tensor and its first k derivatives are bounded, that is, tend to a C^0 (or C^{0-}) function. Alternatively, it may be a C^k (or C^{k-}) *scalar curvature singularity* or a *scalar polynomial curvature singularity* if some scalar does not tend to a C^0 (or C^{0-}) function.

The most familiar class of singularities are the scalar curvature singularities. These include the 'big bang' and 'black hole' types of singularity which closely correspond to one's intuitive concept of a real physical singularity. As such a singularity is approached, some physical quantities diverge and all observers feel unbounded tidal forces.

The non-scalar curvature and quasiregular singularities are much less well understood and have been less fully investigated. Consider, for example, a space-time with a non-scalar curvature singularity. No curvature scalars diverge in this case, yet some components of the Riemann tensor evaluated in a PPON frame along an incomplete curve do not tend to

finite limits as the singularity is approached. The physical effect of this is that all test particles which fall into the non-scalar curvature singularity feel infinite tidal forces, but observers can move arbitrarily close to the singularity on other curves and feel no untoward effects.

Finally, consider a space-time with a quasiregular singularity. In all reasonable frames the Riemann tensor is completely finite. In this case, observers near a quasiregular singularity, including those who fall into the singularity itself, do not feel unbounded tidal forces.

All three types of singularity are found in colliding plane wave solutions. In the following sections and chapters the singularity and global structure of particular solutions will be analysed in more detail.

8.2 The singularity in region IV

In the exact solution of Khan and Penrose (1971) described in Chapter 3, there is a scalar curvature singularity in region IV on the hypersurface $f+g=0$. However, it is not clear whether or not this type of singularity will occur in other solutions, particularly as the Khan–Penrose solution contains impulsive gravitational waves.

This question has been considered by Szekeres (1972), who found that, for colliding gravitational waves with aligned linear polarization, a curvature singularity is inevitable. This work was generalized by Sbytov (1976) to plane gravitational waves with arbitrary polarization, with the same result. However, more recently, counterexamples to these conclusions have been obtained, in the non-aligned case by Chandrasekhar and Xanthopoulos (1986*c*), and in the aligned case by a degenerate solution of Ferrari and Ibañez (1987*b*) and by the algebraically general solutions of Feinstein and Ibañez (1989). In these cases, the curvature scalars remain bounded on the hypersurface $f+g=0$, and scalar polynomial curvature singularities occur in the extensions of the solution through the focusing singularity. These exceptional cases will be considered in detail later on. For the present, it may simply be observed that Szekeres and Sbytov had omitted to include these cases.

Another counterexample proposed by Stoyanov (1979) has proved to be incorrect as it does not satisfy the required boundary conditions (see Section 10.2).

The same question applied to colliding electromagnetic waves is more complicated, and the answers are less conclusive. Generally, one may expect that, for a combination of gravitational and electromagnetic waves, a curvature singularity in region IV will usually occur. However, for purely electromagnetic waves it is less clear. In the first exact solution of this

type, given by Bell and Szekeres (1974), the singularity on the hypersurface $f+g=0$ was shown to be only a coordinate singularity that could easily be removed by a coordinate transformation. This solution, which is conformally flat in the interaction region, will be described in Chapter 14 together with the full analysis of its singularity structure as given by Clarke and Hayward (1989). Other type D electrovac solutions have been obtained by Chandrasekhar and Xanthopoulos (1987*a*) and Papacostas and Xanthopoulos (1988). Further algebraically general solutions without curvature singularities can easily be constructed as will be indicated in Section 17.2. All these have quasiregular singularities that are interpreted as Cauchy horizons on the surface $f+g=0$. These will be described later.

We may conclude that, in all cases, the opposing waves mutually focus each other onto the hypersurface $f+g=0$, on which the contraction of the waves is unbounded and the line element (6.20) is singular. Usually this will be a scalar polynomial curvature singularity, but a large class of of significant exceptions occurs.

It is convenient to point out at this stage that, for colliding gravitational and electromagnetic waves, two of the scalar polynomial invariants (Penrose and Rindler, 1986) are given by

$$I = 2\Psi_0\Psi_4 + 6\Psi_2^2, \qquad J = 6(\Psi_0\Psi_4 - \Psi_2^2)\Psi_2. \tag{8.2}$$

It follows from this that, in order to prove the existence of a scalar polynomial curvature singularity, it is sufficient merely to show that the component Ψ_2 is unbounded. In this case either I or J must be unbounded, and a curvature singularity occurs.

It would, of course, be useful to have a number of general theorems that could be used to determine the singularity structure of particular classes of solutions. Apart from the earlier results of Szekeres (1972) and Sbytov (1976), the only general theorem to date is that of Tipler (1980). This is in fact a straightforward generalization of a theorem of Penrose (1965*b*) (see also Hawking and Ellis 1973, p.263). Quoting it directly:

> **Theorem 8.1** (Tipler) *Let (M,g) be a space-time with g at least C^2, and suppose (M,g) has two globally defined commuting space-like Killing vector fields ∂_x and ∂_y, which together generate plane symmetry. If, (1) the null convergence condition holds; (2) at least one of the six quantities Ψ_0, Ψ_4, Φ_{00}, Φ_{22}, σ, λ is non-zero at some point p in (M,g); and (3) through the point p there is a space-like partial Cauchy surface S, which is everywhere tangent to ∂_x and ∂_y, and S is non-compact in the space-like direction normal to ∂_x and ∂_y; then (M,g) is null incomplete.*

This is an interesting theorem which seems to prove the existence of singularities for a large class of colliding plane waves. However, it requires that the metric be at least C^2 everywhere. It thus excludes situations involving impulsive gravitational waves. It does not apply therefore to the solutions of Khan and Penrose (1971), Ferrari and Ibañez (1987*b*), and the exceptional solution of Chandrasekhar and Xanthopoulos (1986*b*). Nor does it apply to the colliding electromagnetic wave solution of Bell and Szekeres (1974) in which impulsive gravitational waves are generated by the collision. In addition, it proves only geodesic incompleteness and says nothing about curvature singularities.

There exists, however, a very large class of exceptional solutions in which curvature singularities do not occur. For gravitational waves these include a degenerate Ferrari–Ibañez (1987*b*) solution and the solution of Chandrasekhar and Xanthopoulos (1986*b*), which are respectively parts of the Schwarzschild and Kerr space-times. These have been described in more detail by Ferrari and Ibañez (1988), and will be described in Sections 10.5 and 13.3. They contain impulsive wave components, and therefore do not satisfy the conditions of Tipler's theorem. In all these solutions, the singularity that occurs when $f+g=0$ corresponds to a Cauchy horizon. This may be followed by either a space-like, or a time-like curvature singularity. The latter case would indicate that, if it were possible for real observers to pass through the horizon, then most would miss this singularity.

A further class of exceptional vacuum solutions in which the curvature singularity is replaced by a Cauchy horizon has been obtained by Feinstein and Ibañez (1989). These solutions contain a subclass in which the approaching waves have smooth wave fronts and the metric is everywhere at least C^2. They thus satisfy the conditions of Tipler's theorem. They are geodesically incomplete, but the focusing hypersurface in this case is only a quasiregular singularity. These solutions thus illustrate the fact that a proof of null incompleteness does not necessarily imply the existence of a scalar polynomial curvature singularity.

The Killing–Cauchy horizons that occur in these solutions have particular significance as the caustics formed by the mutual focussing of the opposing waves. They have also been further investigated by Yurtsever (1987), who has shown that they are unstable against plane-symmetric perturbations. It is therefore reasonable to conclude that the existence of space-like singularities is likely to be a generic feature of colliding plane wave solutions. This conclusion is also supported by the work of Chandrasekhar and Xanthopoulos (1987*b*), who have shown that the presence of an arbitrarily small amount of dust will change a horizon into a curvature singularity.

In these exceptional cases, it is necessary for the approaching waves to take very specific forms in order to achieve the appropriate solution in region IV. The instability of the horizons in these solutions can easily be demonstrated in that they require very specific forms for the initial functions $f(u)$ and $g(v)$. Any slight variation in these functions would change the Cauchy horizon into a curvature singularity.

Most of the exceptional solutions that have been explicitly obtained are of algebraic type D, although the solutions of Feinstein and Ibañez are algebraically general. Perhaps it should also be remarked that algebraic type D solutions do not necessarily have horizons rather than singularities, a counterexample being one of the degenerate Ferrari–Ibañez (1987*b*) solutions.

An alternative proof of Tipler's Theorem 8.1 has been given by Yurtsever (1988*b*). This emphasizes the role and necessity of the assumption of a strict plane symmetry. In this approach, the exceptional solutions that have Killing–Cauchy horizons in the interaction region do not satisfy the condition of strict plane symmetry as defined by Yurtsever.

The structure of the singularity in the interaction region has been further analysed by Yurtsever (1988*c*, 1989*a*), who has shown that the metric is asymptotic to an inhomogeneous Kasner solution as the singularity is approached. Initially (Yurtsever 1988*c*), he considered the case when the approaching waves have constant aligned polarization and obtained explicit expressions which relate the asymptotic Kasner exponents along the singularity to the initial data posed along the wave fronts of the approaching waves. From these expressions it is clear that, for specific choices of initial data, the curvature singularity formed by the interacting waves degenerates to a coordinate singularity. It can also be concluded that these Killing–Cauchy horizons are unstable against small but generic perturbations of the initial data and that, in a very precise sense, 'generic' initial data always produce all-embracing space-like curvature singularities. In the subsequent paper (Yurtsever, 1989*a*), he has shown that these same conclusions are also reached in the case when the polarization of the approaching waves is arbitrary.

8.3 The Khan–Penrose solution

The solution of Khan and Penrose (1971) which has already been discussed in detail in Chapter 3, may now be reconsidered. It was shown there in particular, that the scalar invariants are unbounded on the hypersurface $f+g=0$ in region IV, where $f=\frac{1}{2}-u^2$ and $g=\frac{1}{2}-v^2$. This clearly demonstrates the existence of a scalar curvature singularity on this hypersurface. In addition to this, there are apparent coordinate singu-

larities extending from it into regions II and III. The naïve singularity structure of this solution is thus as represented in Figure 3.2.

It is now necessary to consider in more detail the character of the apparent singularities in these initial regions II and III. As described in Section 3.3, if it is possible for real particles to pass through them, then it will be possible for those particles to subsequently look back and observe the naked singularity in region IV. If correct, this would provide a counterexample to the cosmic censorship hypothesis. It is more likely, however, that these singularities will prove to be more than artificial coordinate singularities that particles can pass through.

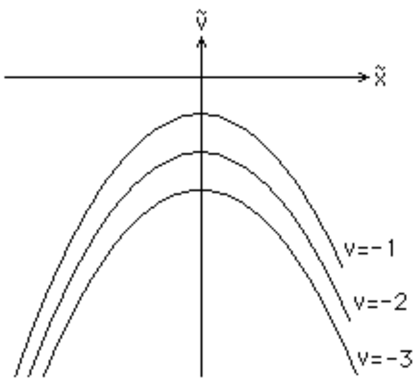


Figure 8.1 Sections through surfaces v = constant, $y = 0$, when $u = 0$, for three different (negative) values of v.

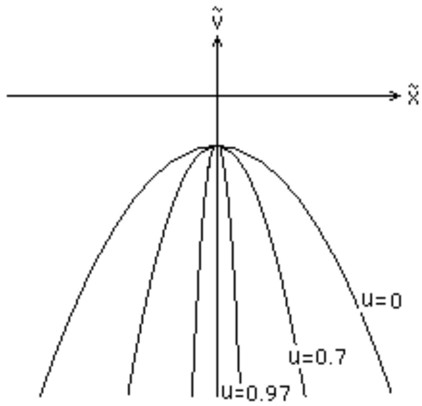


Figure 8.2 Sections through the surface v = constant, $y = 0$, for three different values of u.

This question has been discussed in an interesting paper by Matzner and Tipler (1984). Because of the obvious symmetry, we may concentrate on the singularity in region II. The first point to note is that, since the initial waves are impulsive, the interiors of both regions I and II are flat. For $v \leq 0$, curvature only occurs on the null boundary $u = 0$. The line elements for regions I and II are respectively

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \tag{8.3}$$

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1-u)^2\mathrm{d}x^2 - (1+u)^2\mathrm{d}y^2. \tag{8.4}$$

That these both describe flat space-time can be demonstrated by transforming (8.4) to the form (8.3) by putting

$$u = \tilde{u}, \qquad v = \tilde{v} + \frac{\frac{1}{2}\tilde{x}^2}{1-\tilde{u}} - \frac{\frac{1}{2}\tilde{y}^2}{1+\tilde{u}}, \qquad x = \frac{\tilde{x}}{1-\tilde{u}}, \qquad y = \frac{\tilde{y}}{1+\tilde{u}}. \tag{8.5}$$

It may be observed that the coordinate singularity $u = 1$ is effectively removed by this transformation, and there appears no *a priori* reason preventing the continuation of the coordinates $\tilde{x}^\mu$ through it.

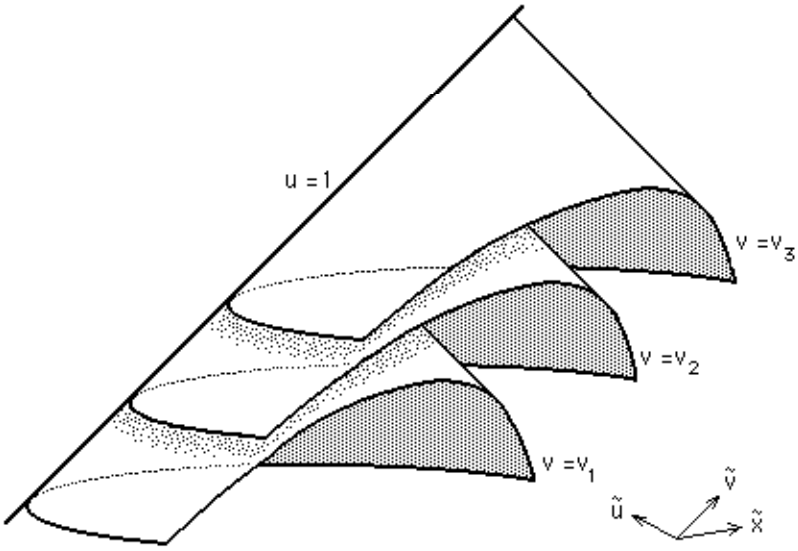


Figure 8.3 Surfaces $v =$ constant, $y = 0$, $0 < u < 1$, for different (negative) values of v. Notice that the line $u = 1$ is common to all surfaces.

Matzner and Tipler proceed to investigate the properties of the null hypersurfaces $v =$ constant, working with the null Minkowski coordinates $\tilde{x}^\mu$. Projections of these hypersurfaces are illustrated in Figures 8.1, 8.2 and 8.3. It can be seen that as $u \to 1$ the curvature of these hypersurfaces diverges. It should also be noticed that the surfaces $v =$ constant, considered as embedded in the three-space $\tilde{y} = 0$, consist of nested two-surfaces all having the line $u = 1$ in common.

It is then possible show that the hypersurface $u = 1$ is not merely a coordinate singularity, but is actually a singularity of space-time in the sense that there does not exist a C^1 extension from region II to this surface. This can not be a curvature singularity, since the curvature tensor on it is zero. Matzner and Tipler accordingly describe it as a 'fold singularity'. With this interpretation, the structure of the Khan–Penrose solution is thus as described in Figure 8.4.

This result raises another problem. In the absence of the second wave, the entire space-time may be described by (8.3) for $u < 0$, and by (8.4) for $u \geq 0$, for all values of v. In this case, $u = 1$ is merely a coordinate singularity. It has no physical significance and can be removed by a coordinate transformation. However, once the second wave is present,

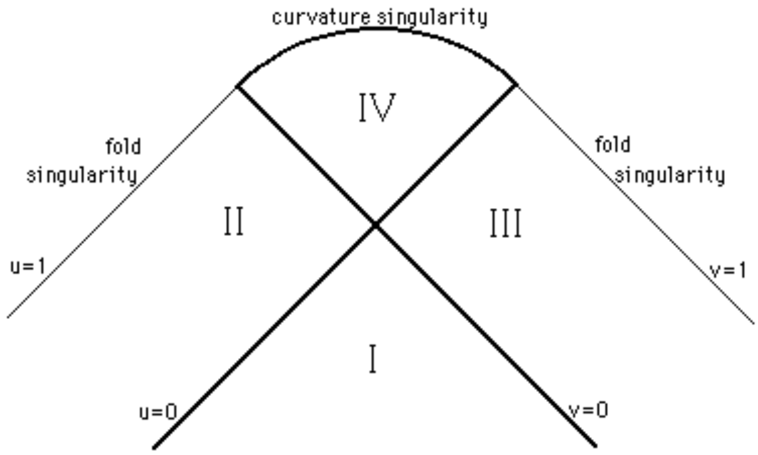


Figure 8.4 The singularity structure of the Khan–Penrose solution.

with wavefront $v = 0$, the above result states that $u = 1$ becomes a space-time singularity even for $v < 0$. This seems to violate our familiar concept of causality. The presence of the second wave seems to change the character of the prior singularity.

The resolution of this problem can be demonstrated using Figure 8.3. The family of surfaces for $v =$ constant can be continued up to $v = 0$. However, there is a curvature singularity at $u = 1$, $v = 0$. The final surface $v = 0$, which forms a bound for the prior surfaces, therefore contains a singularity on the line $u = 1$. In fact, this line $u = 1$, $v = 0$ is actually common to the entire family of surfaces for which $v \leq 0$. Thus the line $u = 1$, $v \leq 0$ must be considered as a singularity of space-time, which is identified with the point $u = 1$, $v = 0$. In the Khan–Penrose solution, the singularities in regions II and III are topological singularities that are identified with the curvature singularity in region IV by the particular choice of coordinates.

This point may be further clarified by considering a family of null geodesics that are initially parallel in region I, and enter region II. The null geodesic $x = y = 0$, $v =$ constant, starts in region I, passes through the gravitational wave and apparently ends at the $u = 1$ singularity. This may be contrasted, however, with a neighbouring geodesic which is given using the the null Minkowski coordinates $\tilde{x}^\mu$ defined by (8.5), by $\tilde{x} = \epsilon$, $\tilde{y} = 0$, $\tilde{v} =$ constant $= \tilde{v}_0$. This geodesic is given by

$$x = \frac{\epsilon}{1-u}, \qquad y = 0, \qquad v = \tilde{v}_0 + \frac{\frac{1}{2}\epsilon^2}{1-u}. \tag{8.6}$$

From this it is clear that v increases indefinitely as $u \to 1$ even for arbitrarily small values of ϵ. Thus, although the geodesic with $x = y = 0$

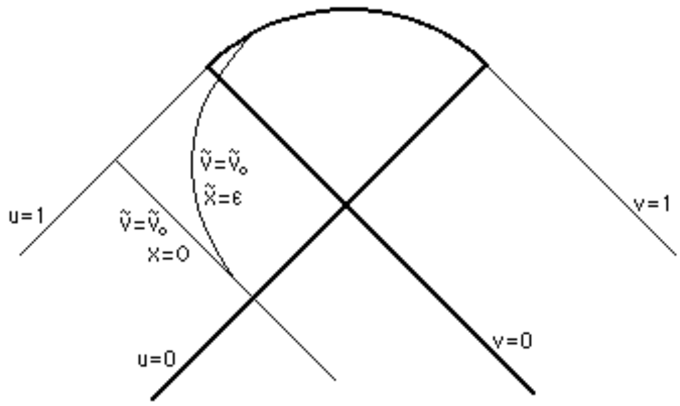


Figure 8.5 Projections of two neighbouring null geodesics onto the plane $x = 0$, $y = 0$. The geodesics are parallel and arbitrarily close in region II, which is flat.

approaches the 'fold singularity' apparently at a finite distance from the curvature singularity in region IV, an arbitrarily close geodesic that is initially parallel to it in region II diverges from it and crosses into region IV before it reaches the hypersurface $u = 1$. This geodesic subsequently terminates in the curvature singularity in region IV. The projection of these two geodesics onto the plane $x = y = 0$ is illustrated in Figure 8.5.

It has thus been argued that the singularities $u = 1$ and $v = 1$ in regions II and III are essentially extensions of the singularity $f + g = 0$ in region IV. The apparent non-causality that appears in the introduction of these singularities is a consequence of the projection of the space-time onto the plane $x = y = 0$. Such a projection does not in general preserve causal relations.

Although Figure 8.4 is very useful in formulating the colliding plane wave problem by dividing the space-time up into appropriate regions, in many ways it is misleading. In particular, it does not adequately describe the singularity structure of the solution, which appears to be non-causal. The problem arises from the attempt to represent space-time on a two-dimensional diagram. In many ways it would be preferable to attempt to picture a three-dimensional structure as in Figure 8.3. Such a picture has been drawn by Penrose, and was included in the paper of Matzner and Tipler (1984). This structure is illustrated in Figure 8.6 where the four regions are separately represented. Since curvature occurs only on the boundaries of regions I, II and III, the differently shaped boundaries of these regions have to be identified artificially. Region IV is curved and no adequate representation of it can be given.

Using the classification scheme of Ellis and Schmidt (1977), it can be seen that the singularity in region IV of the Khan–Penrose solution is

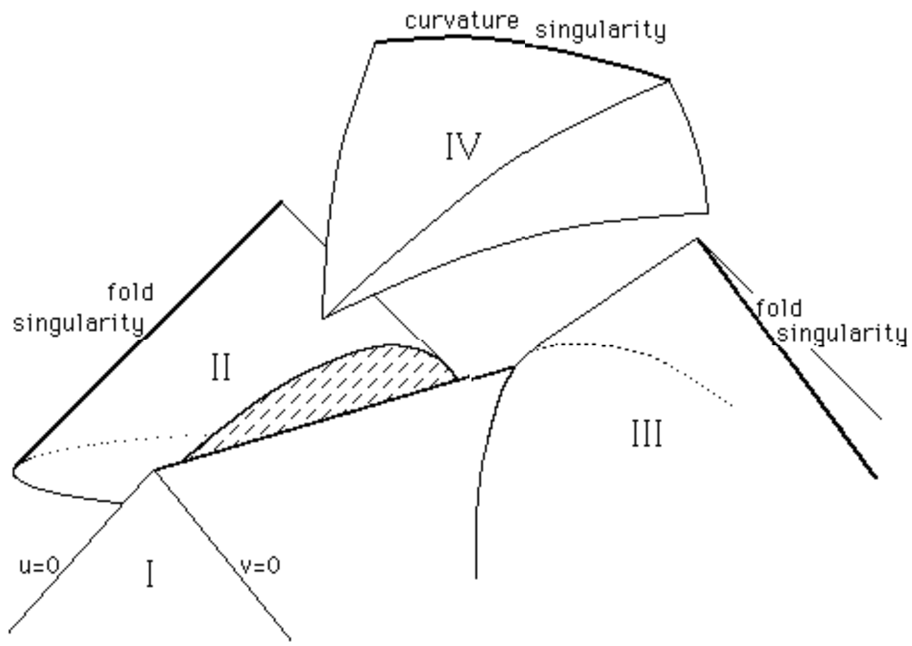


Figure 8.6 A three-dimensional picture of the Khan–Penrose solution with $y = 0$. The interiors of regions I, II and III are flat, but the boundaries have intrinsically different geometry. Points on opposite sides of the wavefronts $u = 0$ and $v = 0$ must be identified. Region IV is curved, so its representation should be considered no more than schematic.

clearly a scalar polynomial curvature singularity. Also the fold singularities in regions II and III that have a topological character, are quasiregular singularities since on them the curvature tensor is zero.

8.4 The structure of other solutions

In the previous section the singularity structure of the Khan–Penrose solution has been analysed in some detail. This has been possible because the interiors of regions I, II and III are all flat. In the general problem, however, it is only the background region I that is taken to be flat, and regions II and III, as well as IV, are curved. Consequently it is not possible in general to analyse the singularity structure in such detail.

It is reasonable to assume, however, that most colliding plane wave solutions will have the same general singularity structure as that of the Khan–Penrose solution. This in fact turns out to be the case, as will be shown here and in the following chapters. In almost all cases a curvature singularity develops in region IV on the hypersurface $f + g = 0$, although there is a large class of exceptional solutions in which the singularity is replaced by a Cauchy horizon. These exceptional solutions provide the

only significant variation of the general singularity structure described above. The structure of these solutions will be described as they are derived in the following chapters.

It has also been shown that, for all colliding plane wave solutions, coordinate singularities necessarily occur in regions II and III. For vacuum solutions, in these regions at most one component, either Ψ_4 or Ψ_0, of the curvature tensor is non-zero. Scalar polynomial curvature singularities therefore cannot occur. The coordinate singularities on the hypersurfaces $f = -\frac{1}{2}$ and $g = -\frac{1}{2}$ may thus either be quasiregular singularities if the curvature tensor on these surfaces is bounded, or they must be non-scalar curvature singularities if the curvature components become unbounded.

It will now be argued that the topological 'fold' singularities of regions II and III in the Khan–Penrose solution, are also general features of all colliding plane wave solutions.

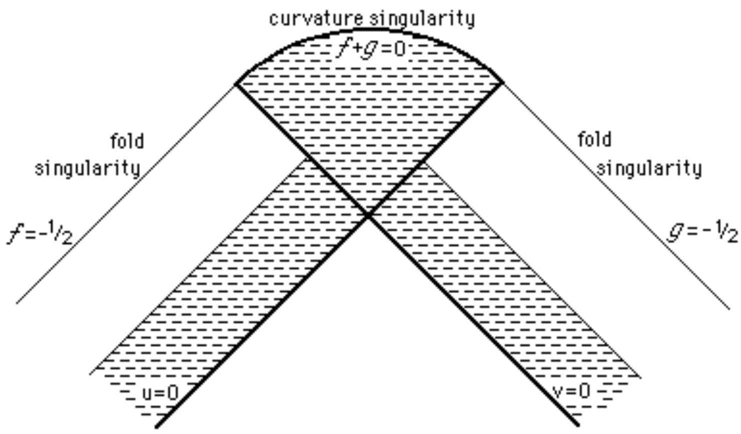


Figure 8.7 The singularity structure for colliding sandwich gravitational waves. The shaded regions have non-zero curvature.

If the approaching waves are 'sandwich' gravitational waves, then the regions behind the waves are flat. These flat regions must be described by metrics that are equivalent to those of the Khan–Penrose solution for regions II and III. After all, the impulsive waves considered by Khan and Penrose may be regarded as idealizations of such waves. In these cases similar 'fold' singularities will occur, as described in Figure 8.7.

These singularities in regions II and III have been more thoroughly investigated by Konkowski and Helliwell (1989). By concentrating on the Szekeres (1972) family of solutions, they have shown that these are quasiregular singularities in the classification scheme of Ellis and Schmidt (1977) in the cases of impulsive and sandwich waves. In the alternative case of thick gravitational waves, they are non-scalar curvature singularities. In addition, by considering scalar wave perturbations in the impul-

sive wave case, they have also shown that the quasiregular singularities are unstable and convert to scalar curvature singularities.

To consider further the general character of these coordinate singularities in regions II and III we may concentrate on region II. The metric in this region may be taken to be of the form (6.20), but with the metric functions U, V, W and M all depending on u only. These functions may be obtained from those in region I simply by replacing g by $\frac{1}{2}$, or v by 0. Considering the geodesics in this region, it is clear that there exist three conserved momentum components p_x, p_y and p_v besides the energy integral. These are given by

$$\begin{aligned} p_x &= -2e^{-U}(e^V \cosh W\dot{x} - \sinh W\dot{y}) \\ p_y &= -2e^{-U}(e^{-V} \cosh W\dot{y} - \sinh W\dot{x}) \\ p_v &= 2e^{-M}\dot{u} \\ \epsilon &= 2e^{-M}\dot{u}\dot{v} - e^{-U}\left(e^V \cosh W\dot{x}^2 - 2\sinh W\dot{x}\dot{y} + e^{-V}\cosh W\dot{y}^2\right) \\ &= p_v\dot{v} - \frac{e^{-U}}{4}\left(e^{-V}\cosh W p_x^2 + 2\sinh W p_x p_y + e^V \cosh W p_y^2\right) \end{aligned} \tag{8.7}$$

where ϵ may be taken to be 0 on null geodesics and 1 on time-like geodesics.

Using (8.7), we obtain the equation

$$\begin{aligned} p_v^2\frac{\mathrm{d}v}{\mathrm{d}u} = {} & \frac{e^{-(U+M)}}{2}\left(e^{-V}\cosh W p_x^2 + 2\sinh W p_x p_y + e^V \cosh W p_y^2\right) \\ & + 2\epsilon\, e^{-M}. \end{aligned} \tag{8.8}$$

Expressions for U and M in region II can be obtained from (7.2) and (7.8) in the form

$$e^{-U} = (\tfrac{1}{2} + f), \qquad e^{-M} = -\frac{f'}{\sqrt{\frac{1}{2}+f}}e^{-S_1(f)}. \tag{8.9}$$

With these, (8.8) may be integrated to give

$$\begin{aligned} p_v^2 v = -\int\Bigg(& \frac{e^{-S_1}}{2(\frac{1}{2}+f)^{3/2}}\left(e^{-V}\cosh W p_x^2 + 2\sinh W p_x p_y + e^V \cosh W p_y^2\right) \\ & + 2\epsilon\frac{e^{-S_1}}{\sqrt{\frac{1}{2}+f}}\Bigg)\mathrm{d}f. \end{aligned} \tag{8.10}$$

It may immediately be seen from this that no time-like or null geodesics with either p_x or p_y non-zero can avoid crossing the surface $v = 0$ into

region IV before reaching the surface on which $f=-\frac{1}{2}$. The only exception occurs when $p_x=p_y=0$. This clearly manifests the character of the fold singularity as described in the previous section.

In colliding plane wave problems, what is essentially a coordinate singularity for a single wave is transformed into a fold singularity prior to the collision by the presence of curvature singularities at the points $v=0$, $f=-\frac{1}{2}$, and $u=0$, $g=-\frac{1}{2}$. Generally these are associated with a space-like singularity on $f+g=0$, but as described above, there are exceptional cases in which this singularity is replaced by a horizon. However, even in these cases, it is found that there exist at least distribution valued singularities just at these points. From this, it may be argued that the presence of fold singularities in regions II and III appears to be a general feature of colliding plane wave problems.

It has thus been argued that the general structure of all colliding plane wave solutions is as illustrated in Figure 8.6 with $y=0$ with the possible exception that, for some solutions, the curvature singularity in region IV is replaced by a Cauchy horizon.

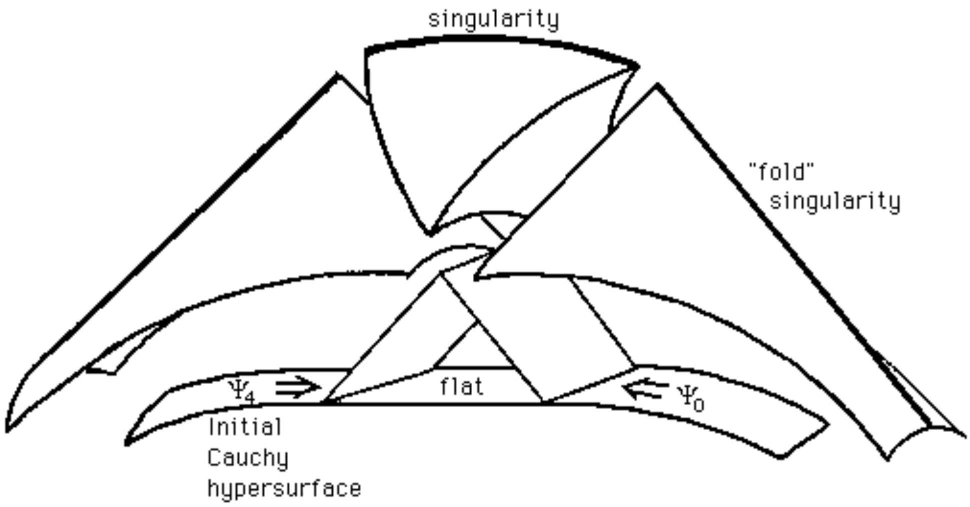


Figure 8.8 The general structure of colliding plane wave solutions related to the initial hypersurface on which initial Cauchy data is set.

It is also appropriate to attempt to relate this general singularity structure to some initial hypersurface on which the initial Cauchy data for the colliding wave problem is specified. To do this it is necessary to incorporate the properties of plane waves that have been described in Section 4.4 and illustrated in Figure 4.1. In particular, it must be noted that no global space-like hypersurface exists on which initial Cauchy data for the problem can be set. The existence of the topological 'fold' singularities in regions II and III implies bounds for the initial Cauchy

hypersurface in both of the directions from which the waves emerge. This property is illustrated in Figure 8.8.

It can immediately be deduced from Figure 8.8 that, for solutions in which the curvature singularity in region IV is replaced by a Cauchy horizon, any possible extension through this horizon must be non-unique. Any extended space-time beyond this singularity will depend on extra initial data in addition to that specified on the initial Cauchy hypersurface for the colliding wave problem.

The above arguments lead to the conclusion that the usual initial data for colliding plane waves leads to a unique solution only up to the topological singularities $f=-\frac{1}{2}$ and $g=-\frac{1}{2}$ in regions II and III, and to the 'focusing' singularity $f+g=0$ in the interaction region. This focusing singularity is normally a curvature singularity, but this may be replaced by a Cauchy horizon. In this case, any future extension through the horizon is non-unique.

9

THE SZEKERES CLASS OF VACUUM SOLUTIONS

The first exact solution to be published which describes a collision between plane waves was in fact that of Szekeres (1970). This describes the collision of plane gravitational waves with step wavefronts. The solution discussed in Chapter 3, which describes the collision between impulsive gravitational waves, was published a little later by Khan and Penrose (1971). Following this, a substantial paper in which the subject is analysed in detail was produced by Szekeres (1972). This paper includes a derivation of the field equations in the form given in Chapter 6, a general class of exact solutions which includes the two mentioned above as special cases, and a discussion of the singularities that arise. The purpose of this chapter is to discuss the properties of this general class of solutions.

9.1 The solution in region IV

The vacuum field equations appropriate to region IV of the colliding plane wave problem may be taken here in the form of equations (6.22*a*–*f*). These are second order differential equations for the four metric functions $U(u,v)$, $V(u,v)$, $W(u,v)$ and $M(u,v)$.

Equation (6.22*a*) may immediately be integrated as in (6.24) to give

$$e^{-U} = f + g \tag{9.1}$$

where $f = f(u)$ and $g = g(v)$ are arbitrary decreasing functions which, according to the appropriate boundary conditions (7.3), are required to satisfy $f(0) = \frac{1}{2}$, $f'(0) = 0$ and $g(0) = \frac{1}{2}$, $g'(0) = 0$.

In the Szekeres class of solutions, the approaching waves have constant aligned polarization. In these solutions therefore

$$W = 0, \tag{9.2}$$

and the main equations (6.22*d*, *e*) reduce to the single equation

$$2V_{uv} = U_u V_v + U_v V_u. \tag{9.3}$$

For this, Szekeres has obtained the solution

$$V = -2k_1 \tanh^{-1}\left(\frac{\frac{1}{2}-f}{\frac{1}{2}+g}\right)^{1/2} - 2k_2 \tanh^{-1}\left(\frac{\frac{1}{2}-g}{\frac{1}{2}+f}\right)^{1/2} \tag{9.4}$$

which contains two arbitrary constants[1] k_1 and k_2. The expression (9.4) may also be written in the alternative form

$$e^V = \left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f}}\right)^{k_1} \left(\frac{\sqrt{\frac{1}{2}+f}-\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f}+\sqrt{\frac{1}{2}-g}}\right)^{k_2} . \tag{9.5}$$

With this expression for V, the remaining equations in (6.22) may be integrated to give

$$\begin{aligned} M = &-\log(cf'g') - \tfrac{(k_1^2+k_2^2+2k_1k_2-1)}{2}\log(f+g) \\ &+ \tfrac{k_1^2}{2}\log(\tfrac{1}{2}-f) + \tfrac{k_2^2}{2}\log(\tfrac{1}{2}+f) + \tfrac{k_2^2}{2}\log(\tfrac{1}{2}-g) + \tfrac{k_1^2}{2}\log(\tfrac{1}{2}+g) \\ &+ 2k_1k_2\log\left(\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}-g} + \sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}+g}\right) \end{aligned} \tag{9.6}$$

where c is a constant.

As anticipated in the discussion in Section 7.2, it may be observed that this expression contains the necessary multiples of $\log(\frac{1}{2}-f)$ and $\log(\frac{1}{2}-g)$ that are required to cancel the effects of the unbounded term $\log f'g'$ on the boundary. If the leading terms in the expansions for f and g are

$$f = \tfrac{1}{2} - (c_1u)^{n_1} + \ldots, \qquad g = \tfrac{1}{2} - (c_2v)^{n_2} + \ldots \tag{9.7}$$

where $n_i \geq 2$, $i = 1, 2$, then e^{-M} is continuous across the boundaries if

$${k_1}^2 = 2(1 - 1/n_1), \qquad {k_2}^2 = 2(1 - 1/n_2). \tag{9.8}$$

It may thus be observed that the constants k_1 and k_2 are restricted to the range satisfying

$$1 \leq {k_i}^2 < 2, \qquad i = 1, 2. \tag{9.9}$$

[1] It may be noticed that the constants used here are minus one half of those used by Szekeres.

It is also appropriate to choose

$$c = (c_1 n_1 c_2 n_2)^{-1} \tag{9.10}$$

to achieve the usual flat metric (3.6) in region I.

When evaluating the components of the Weyl tensor, it is convenient to introduce the new function

$$H = 2k_1 \sqrt{\tfrac{1}{2} - f}\sqrt{\tfrac{1}{2} + g} - 2k_2 \sqrt{\tfrac{1}{2} - g}\sqrt{\tfrac{1}{2} + f}. \tag{9.11}$$

With this, it can be seen that

$$V_u = -\frac{f'}{f+g} H_f, \qquad V_v = \frac{g'}{f+g} H_g, \tag{9.12}$$

and the non-zero Weyl tensor components are

$$\begin{aligned}
\Psi_0^\circ &= -\frac{g'^2}{2(f+g)} H_{gg} - \frac{g'^2}{4(f+g)^2} H_g (1 - H_g^2) \\
\Psi_2^\circ &= -\frac{f'g'}{4(f+g)^2} (1 - H_f H_g) \\
\Psi_4^\circ &= \frac{f'^2}{2(f+g)} H_{ff} + \frac{f'^2}{4(f+g)^2} H_f (1 - H_f^2).
\end{aligned} \tag{9.13}$$

In his paper, Szekeres chooses

$$f = \tfrac{1}{2} - (c_1 u)^{n_1} \qquad \text{and} \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} \tag{9.14}$$

exactly. In this case, the terms $\frac{k_1^2}{2}\log(\frac{1}{2} - f)$ and $\frac{k_2^2}{2}\log(\frac{1}{2} - g)$ in (9.6) exactly cancel the term $\log(cf'g')$, and M is given by

$$e^{-M} = \frac{(f+g)^{(k_1^2 + k_2^2 + 2k_1k_2 - 1)/2}}{(\frac{1}{2} + f)^{k_2^2/2} (\frac{1}{2} + g)^{k_1^2/2} \left(\sqrt{\frac{1}{2} - f}\sqrt{\frac{1}{2} - g} + \sqrt{\frac{1}{2} + f}\sqrt{\frac{1}{2} + g}\right)^{2k_1k_2}}. \tag{9.15}$$

This form is still completely general since it has, in effect, simply used the coordinate freedom (6.7). However, it is sometimes convenient to retain this freedom, so this restriction will not be made in the following section.

It may now be pointed out that, with the restriction (9.14), the above solution includes the Khan–Penrose (1971) solution for colliding impulsive waves when $n_1 = n_2 = 2$, $k_1 = k_2 = 1$. It also includes the Szekeres (1970) solution for colliding step waves when $n_1 = n_2 = 4$, $k_1 = k_2 = \sqrt{3/2}$.

9.2 The approaching waves

Having obtained an exact solution in region IV, the question now is to find the initial conditions which give rise to it. In fact, it is quite simple to use the method described in Section 7.2 to extend any solution in region IV back into regions II, III and I. We obtain the corresponding solutions in regions II and III simply by replacing g by $\frac{1}{2}$ and f by $\frac{1}{2}$ alternately. In this section we will concentrate on region II, and put $U = -\log\sqrt{\frac{1}{2}+f}$. The equivalent solution in region III can be obtained from this by replacing f by g, and by interchanging u and v.

Using the suggested method, it can be seen that the solution in region II must have the line element

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - (\tfrac{1}{2}+f)\left(e^V\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right) \tag{9.16}$$

where, retaining the coordinate freedom in u,

$$\begin{aligned} f &= \tfrac{1}{2} - (c_1u)^{n_1} + \dots \\ e^V &= \left(\frac{1-\sqrt{\frac{1}{2}-f}}{1+\sqrt{\frac{1}{2}-f}}\right)^{k_1}, \qquad\qquad \frac{k_1^2}{2} = 1 - \frac{1}{n_1} \\ e^{-M} &= -\frac{f'(\frac{1}{2}+f)^{(k_1^2-1)/2}}{c_1n_1(\frac{1}{2}-f)^{k_1^2/2}}. \end{aligned} \tag{9.17}$$

In order to interpret this solution as a plane wave, it is appropriate to transform the metric to the Brinkmann form

$$\mathrm{d}s^2 = 2\mathrm{d}\tilde{u}\mathrm{d}r + h(\tilde{u})(\tilde{x}^2-\tilde{y}^2)\mathrm{d}\tilde{u}^2 - \mathrm{d}\tilde{x}^2 - \mathrm{d}\tilde{y}^2 \tag{9.18}$$

by putting

$$\begin{aligned} \tilde{x} &= \sqrt{\tfrac{1}{2}+f}\ e^{V/2}x \\ \tilde{y} &= \sqrt{\tfrac{1}{2}+f}\ e^{-V/2}x \\ r &= u - \tfrac{1}{4}(U_u - V_u)e^M\tilde{x}^2 - \tfrac{1}{4}(U_u+V_u)e^M\tilde{y}^2 \\ \tilde{u} &= -\int\frac{(\frac{1}{2}+f)^{(k_1^2-1)/2}}{c_1n_1(\frac{1}{2}-f)^{k_1^2/2}}\frac{\mathrm{d}f}{\mathrm{d}u}\mathrm{d}u. \end{aligned} \tag{9.19}$$

The wave profile in equation (9.18) now takes either of the forms

$$h(\tilde{u}) = c_1\delta(\tilde{u}) \qquad \text{if} \quad n_1 = 2, \quad k_1 = 1 \tag{9.20}$$

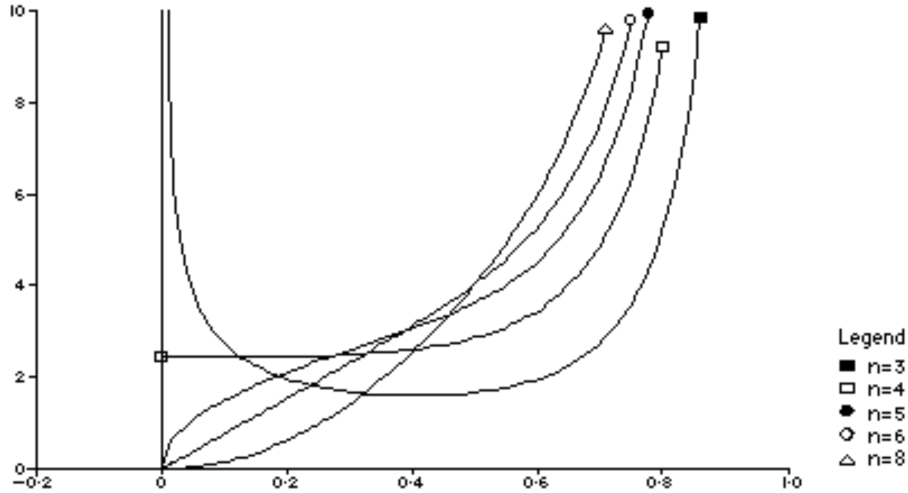


Figure 9.1 Some wave profiles for the approaching waves in the Szekeres solution with $f = 1/2 - u^n$. The wavefront is unbounded if $2 < n < 4$. It contains a step if $n = 4$, and is smooth if $n > 6$. All profiles become unbounded as u approaches 1.

or[2]

$$h(\tilde{u}) = c_1^2 n_1^2 \frac{k_1}{4}\left(1 - \frac{2}{n_1}\right)\frac{(\frac{1}{2} - f)^{1/2-2/n_1}}{(\frac{1}{2} + f)^{3-2/n_1}}\Theta(\tilde{u}) \qquad \text{if} \quad n_1 > 2. \tag{9.21}$$

Particular cases in which $c_1 = 1$ and $f = \frac{1}{2} - u^{n_1}$ are illustrated in Figure 9.1 for $n_1 = 3, 4, 5, 6$ and 8.

It may be recalled from (4.5) that $\Psi_4 = h(\tilde{u})$. Thus, it can be clearly seen that the approaching gravitational wave in region II is an impulsive wave if $n_1 = 2$, has an unbounded wavefront if $2 < n_1 < 4$, has a step wavefront if $n_1 = 4$, has a continuous wavefront if $n_1 > 4$, and has a smooth wavefront if $n_1 > 6$. It may also be observed that, if $n_1 \geq 4$, Ψ_4 is monotonically increasing.

9.3 The singularity structure

In this family of solutions, there is always a scalar polynomial curvature singularity in region IV on the surface on which $f + g = 0$. This can be seen by computing the scalar invariant given here by

$$I = 2e^{2M}(3\Psi_2^{\circ 2} + \Psi_0^\circ \Psi_4^\circ). \tag{9.22}$$

[2] It may be noticed that an expression for the profile given by Halil (1979) appears to be incorrect.

Using (9.6) and (9.13), I can be seen to be unbounded when $f+g=0$ for all values of k_1 and k_2 in the required range (9.9). This is similar to the singularity in region IV of the Khan–Penrose solution which, of course, it includes as a special case.

Another singularity occurs in cases in which a wave with an unbounded wave front moves in a region where the opposing wave is non-zero. For example, if $n_2 \neq 2$ and $2 < n_1 < 4$ so that $1 < k_1^2 < \frac{3}{2}$, then I is unbounded on the boundary $u = 0$ between regions III and IV. Similarly, the boundary $v = 0$, $u \geq 0$ is singular if $n_1 > 2$ and $2 < n_2 < 4$ so that $1 < k_2 < \frac{3}{2}$. Apart from cases containing impulsive waves, where $n_1 = 2$ or $n_2 = 2$, the boundaries of region IV are therefore only regular if $n_i \geq 4$ for $i = 1, 2$, which implies the further restriction $\frac{3}{2} \leq k_i^2 < 2$ on (9.9). These singularities, however, are distribution-valued and their physical significance requires further investigation.

The coordinate singularities in regions II and III, however, are not so obviously similar to the fold singularities of the Khan–Penrose solution. The Szekeres solutions are only flat behind the wave front in the special case when $n_1 = n_2 = 2$. In the general case, regions II and III are curved and the analysis given in Section 8.2 cannot be applied.

However, we may notice that in all cases with $n_1 > 2$, Ψ_4 becomes unbounded as $f \to -1/2$ in region II. This clearly indicates that, in these cases, the singularity in this region is a non-scalar curvature singularity. This has been confirmed in rigorous calculations by Konkowski and Helliwell (1989). It has also been argued in the previous chapter that these singularities have similar properties to the fold singularities that occur for colliding impulsive waves as described in Section 8.2 in terms of the behaviour of neighbouring geodesics.

A similar non-scalar curvature singularity also occurs in region III if $n_2 > 2$, as Ψ_0 then becomes unbounded as $g \to -1/2$.

10

OTHER VACUUM SOLUTIONS WITH ALIGNED POLARIZATION

The solutions of Khan and Penrose (1971) and of Szekeres (1970, 1972), discussed in Chapters 3 (and Section 8.2) and 9, describe the collision of plane gravitational waves with aligned linear polarization. It is convenient to consider separately in this chapter a number of other exact solutions that satisfy this same condition.

10.1 A general method

In situations in which the approaching waves are linearly polarized, and their polarization vectors are aligned, it is possible to put $W = 0$ globally. In this case the line element (6.20), and the field equations (6.22*a*–*f*) take a particularly simple form. Equation (6.22*a*) can immediately be integrated to give (6.24) and there is only one main equation, namely (9.3), which is a linear equation in V. For any given solution of this equation, a function M can always be found satisfying the remaining equations (6.22b, c, f), although such a function may not satisfy the required boundary conditions.

The integral (6.24) involves two continuous functions $f(u)$ and $g(v)$ that are monotonically decreasing for positive arguments. As suggested by Szekeres (1972), it is therefore possible to use these as coordinates in region IV, although care has to be taken at the boundaries and it is not possible to extend these coordinates into the prior regions II and III. Using (7.8), the line element in the interaction region can thus be written in the form

$$\mathrm{d}s^2 = \frac{2e^{-S}}{\sqrt{f+g}}\mathrm{d}f\,\mathrm{d}g - (f+g)(e^V\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2), \tag{10.1}$$

and the main equation (9.3) becomes

$$2(f+g)V_{fg} + V_f + V_g = 0. \tag{10.2}$$

which is the well-known Euler–Poisson–Darboux equation.

For any particular solution of (10.2), the new function S can be found by integrating equations (7.9), which now become

$$S_f = -\tfrac{1}{2}(f+g)V_f^2, \qquad S_g = -\tfrac{1}{2}(f+g)V_g^2. \tag{10.3}$$

In order for any particular solution to be appropriate to describe colliding plane waves, the function S must satisfy the conditions described in equations (7.10) to (7.13).

Using this notation, the components of the Weyl tensor are given by the expressions

$$\begin{aligned}
\Psi_0^\circ &= -\frac{g'^2}{4}\left(2V_{gg} + \frac{3}{(f+g)}V_g - (f+g)V_g^3\right) \\
\Psi_2^\circ &= \frac{f'g'}{4}\left(V_fV_g - \frac{1}{(f+g)^2}\right) \\
\Psi_4^\circ &= -\frac{f'^2}{4}\left(2V_{ff} + \frac{3}{(f+g)}V_f - (f+g)V_f^3\right).
\end{aligned} \tag{10.4}$$

It can thus be seen that, although (10.2) is linear so that different solutions for V can be superposed, the associated gravitational waves cannot be simply superposed. In addition, the scale factors A and B satisfying (6.13) and (6.14) must contain components of S, and from (10.3) it can be seen that these components are also non-linear in V.

Once a solution of (10.2) and (10.3) describing the interaction region IV is obtained, the approaching waves in regions II and III can immediately be deduced. For example, in region II, we simply put $g=\frac{1}{2}$ and assume that f takes the form $f=\frac{1}{2}-(c_1u_1)^{n_1}+\dots$. Then, removing the constant from S, it is convenient to put

$$e^{-M(u)} = -\frac{f'}{n_1c_1\sqrt{\frac{1}{2}+f}}e^{-S(u)} \tag{10.5}$$

and the approaching wave is then given by

$$\begin{aligned}
\Psi_4 &= \tfrac{1}{2}n_1c_1V_fe^S\delta(u) \\
&\quad - \tfrac{1}{4}n_1^2c_1^2e^{2S}\left(2(\tfrac{1}{2}+f)V_{ff} + 3V_f - (\tfrac{1}{2}+f)^2V_f^3\right)\Theta(u).
\end{aligned} \tag{10.6}$$

The approaching wave in region III is obtained in exactly the same way in terms of the function $g(v)$.

In order to obtain exact solutions, the first step is to obtain a general class of solutions of (10.2). Szekeres (1972) has indicated how to integrate

this equation using Riemann's method, and has given a general solution expressed as a line integral involving a Legendre function of order $-\frac{1}{2}$. This will be described in Chapter 14. However, because it is very difficult in practice to evaluate the integrals which this method involves, this does not turn out to be a convenient method for obtaining explicit solutions.

As an alternative approach, it may be observed that attempting to solve equation (10.2) by separating the variables leads to the solution

$$V = A(\sigma - f)^{-1/2}(\sigma + g)^{-1/2} \tag{10.7}$$

where A and σ are arbitrary constants. Thus a general class of solutions can be obtained by considering

$$V = \sum_i \frac{A_i}{\sqrt{\sigma_i - f}\sqrt{\sigma_i + g}} \tag{10.8}$$

for arbitrary sequences of constants σ_i and A_i. The particular decomposition (10.8), however, does not turn out to be particularly convenient in the construction of explicit solutions. We will therefore proceed by changing the coordinates.

Before introducing a new set of coordinates, it may be noted in passing that Feinstein and Ibañez (1989) have considered an alternative coordinate system and have expanded a general solution in a different way involving Bessel and Neumann functions of zero order. Their approach will be described in Section 10.7.

The appropriate technique is to consider transformations of the basic functions f and g which are here treated as coordinates. It turns out to be convenient to put

$$f = \tfrac{1}{2}\cos(\psi + \lambda), \qquad g = \tfrac{1}{2}\cos(\psi - \lambda) \tag{10.9}$$

where ψ and λ are considered as time-like and space-like coordinates, which may then be rescaled by putting

$$t = \sin\psi, \qquad z = \sin\lambda. \tag{10.10}$$

With this, it may be noted that

$$f + g = \sqrt{1 - t^2}\sqrt{1 - z^2}, \qquad f - g = -tz \tag{10.11}$$

and t and z can be expressed in terms of f and g by

$$\begin{aligned} t &= \sqrt{\tfrac{1}{2} - f}\sqrt{\tfrac{1}{2} + g} + \sqrt{\tfrac{1}{2} - g}\sqrt{\tfrac{1}{2} + f} \\ z &= \sqrt{\tfrac{1}{2} - f}\sqrt{\tfrac{1}{2} + g} - \sqrt{\tfrac{1}{2} - g}\sqrt{\tfrac{1}{2} + f}. \end{aligned} \tag{10.12}$$

The boundaries of region IV are now the hypersurfaces on which $t = \pm z$, and the focusing hypersurface in this region occurs when $t = 1$. Thus, to correspond to the interaction region, the coordinates must satisfy the inequality $|z| < t \leq 1$.

Using these coordinates, the main equation (10.2) becomes

$$\big((1-t^2)V_t\big)_{,t} - \big((1-z^2)V_z\big)_{,z} = 0. \tag{10.13}$$

A general class of solutions of (10.13) can now be obtained by considering variable separable solutions of the form

$$V = T(t)Z(z). \tag{10.14}$$

With this, (10.13) reduces to the pair of Legendre equations

$$\begin{aligned} (1-t^2)T_{tt} - 2tT_t + n(n+1)T = 0 \\ (1-z^2)Z_{zz} - 2zZ_z + n(n+1)Z = 0 \end{aligned} \tag{10.15}$$

and a class of solutions of (10.13) can be expressed as a sum of products

$$V = \sum_n \big(a_n P_n(t)P_n(z) + q_n Q_n(t)P_n(z) + p_n P_n(t)Q_n(z) + b_n Q_n(t)Q_n(z)\big) \tag{10.16}$$

where $P_n(x)$ and $Q_n(x)$ are Legendre functions of the first and second kinds respectively, and a_n, q_n, p_n and b_n are series of arbitrary constants. In general, of course, Legendre functions of non-integer order may also be included.

The Legendre functions of integer order are well known, but it may still be appropriate to note just the first few.

$$\begin{aligned} &P_0(x) = 1 && Q_0(x) = \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right) \\ &P_1(x) = x && Q_1(x) = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1 \\ &P_2(x) = \tfrac{1}{2}(3x^2-1) && \\ &P_3(x) = \tfrac{1}{2}(5x^3-3x) && Q_2(x) = \tfrac{1}{4}(3x^2-1)\log\left(\frac{1+x}{1-x}\right) - \tfrac{3}{2}x \end{aligned} \tag{10.17}$$

It may easily be shown that the solutions of Khan and Penrose, and of Szekeres are included in this class. The Khan–Penrose solution uses

$$V = -2Q_0(t)P_0(z), \tag{10.18}$$

and the Szekeres solution (9.4) generalizes this to

$$V = -(k_1 + k_2)Q_0(t)P_0(z) - (k_1 - k_2)P_0(t)Q_0(z). \tag{10.19}$$

It may be noted that the Legendre functions of the second kind $Q_n(t)$ are all singular when $t = \pm 1$. The singularity $t = 1$ occurs when $f + g = 0$, and thus can be seen to be associated with the focusing singularity in region IV. In fact, in order to satisfy the boundary conditions, at least one Legendre function of the second kind must be included in the solution of (10.13). This may easily be demonstrated by substituting only the products of Legendre functions of the first kind into the condition (7.15), which can not then be satisfied.

Another solution of (10.13) can be obtained by considering separable solutions involving a sum rather than the product (10.14). This leads to the solution

$$V = -\tfrac{1}{2}a\log(1 - t^2) - \tfrac{1}{2}a\log(1 - z^2) \tag{10.20}$$

where a is an arbitrary constant. This can immediately be seen to be the obvious solution $V = aU$ which, on its own, gives rise to the Kasner metrics as will be shown in the next section. This term may be added to the sum (10.16).

A further solution of (10.2), or (10.13) is given by

$$V = d\cosh^{-1}\left(\frac{c + f - g}{f + g}\right) \tag{10.21}$$

where c and d are arbitrary constants. In fact a series of solutions of this type can be used having different values of c and d. Thus a different representation of the solution of (10.13) can be written in the form

$$V = \sum_i d_i \cosh^{-1}\left(\frac{c_i - tz}{\sqrt{1 - t^2}\sqrt{1 - z^2}}\right). \tag{10.22}$$

Alternatively terms from the sum (10.22) may also be added to those of (10.16). However, as will be shown in (10.68), some of these terms may also be expressed in terms of products of Legendre functions, and so are already included in (10.16). It may be noted in passing that particular terms of this type have been included by Feinstein and Ibañez (1989) in the class of solutions that will be described in Section 10.7.

Having obtained an expression for V as any combination of the terms (10.16), (10.20) and (10.22), it is then necessary to integrate (10.3) to

obtain S. These equations may conveniently be rewritten in terms of the coordinates ψ and λ defined by (10.9) in the form

$$\begin{aligned} S_\psi + S_\lambda &= \frac{\cos\psi\cos\lambda}{2\sin(\psi+\lambda)}(V_\psi + V_\lambda)^2 \\ S_\psi - S_\lambda &= \frac{\cos\psi\cos\lambda}{2\sin(\psi-\lambda)}(V_\psi - V_\lambda)^2. \end{aligned} \tag{10.23}$$

However, V is now so general that no complete integral for S has yet been found. Instead, we proceed by discussing the particular cases that have been obtained.

10.2 The non-singular 'solution' of Stoyanov

We may now consider a paper of Stoyanov (1979) in which he claimed to have obtained a solution without singularities. His method was to look for a regular solution of the field equations in region IV, and then to obtain the global solution simply by requiring that the metric coefficients be continuous across the boundaries.

The solution he presented for region IV has the line element

$$\mathrm{d}s^2 = 2(1 \pm u \pm v)^{(a^2-1)/2}\mathrm{d}u\mathrm{d}v - (1 \pm u \pm v)^{1-a}\mathrm{d}x^2 - (1 \pm u \pm v)^{1+a}\mathrm{d}y^2 \tag{10.24}$$

where a is an arbitrary constant. This clearly uses

$$f = \tfrac{1}{2} \pm u\Theta(u), \qquad g = \tfrac{1}{2} \pm v\Theta(v) \tag{10.25}$$

which does not satisfy the boundary conditions (7.3) which require that f and g must be smooth functions. In fact, it has been shown by Nutku (1981) that the discontinuities in the derivatives of f and g on the boundaries of region IV indicate the presence of an infinite discontinuity in the Ricci tensor on these hypersurfaces. Thus, although (10.24) is a vacuum solution inside region IV, it cannot be a global vacuum solution describing the collision of purely gravitational waves.

It may also be observed that, with the positive signs in (10.24) and (10.25), f and g are increasing functions that are inconsistent with (7.13). Such a possibility can only arise if the impulsive components of the matter tensor occuring on the boundary of region IV have negative energy density. It is in this way that the singularity has been removed. It is only with the possibility of the presence of matter with negative energy density that the focusing effect of colliding waves can be avoided.

Nutku (1981) also pointed out that the line element (10.24) is the well known cosmological solution of Kasner (1921). He gave the explicit coordinate transformation by which the line element (10.24) becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^{2p_1}(\mathrm{d}x^1)^2 - t^{2p_2}(\mathrm{d}x^2)^2 - t^{2p_3}(\mathrm{d}x^3)^2, \tag{10.26}$$

where the coordinate t is not the same as that used elsewhere in this chapter, and

$$p_1 = \frac{2(1-a)}{a^2+3}, \qquad p_2 = \frac{2(1+a)}{a^2+3}, \qquad p_3 = \frac{a^2-1}{a^2+3} \tag{10.27}$$

which clearly satisfies the necessary conditions

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1. \tag{10.28}$$

The Stoyanov solution has been obtained with the solution of (9.3) given by

$$\begin{aligned} V &= aU \\ &= -a\log(f+g) \\ &= -\tfrac{1}{2}a\log(1-t^2) - \tfrac{1}{2}a\log(1-z^2) \end{aligned} \tag{10.29}$$

which is the solution (10.20). With this, the remaining equations in (6.22) may be integrated to give

$$M = -\log(cf'g') + \tfrac{1}{2}(1-a^2)\log(f+g) \tag{10.30}$$

which clearly cannot be continuous across the boundaries of region IV if the junction conditions (7.3) are satisfied.

It must be concluded that the above solution cannot be interpreted in terms of an interaction between plane gravitational waves. Certainly, it is not a counterexample of a solution for colliding waves without singularity. However, in region IV, it is the well known Kasner solution which features regularly in discussions of solution generating techniques. In fact, it turns out that this solution can be used as a ‘seed’ from which other physically acceptable solutions may be derived. It will therefore be referred to again in later sections.

This solution in the case when $a = 1$ has been further investigated by Taub (1988*a*), looking particularly at the properties of the distribution valued curvature tensor.

10.3 The solution of Ferrari and Ibañez and Griffiths

Now consider the case with the solution of (10.11) given by

$$\begin{aligned} V &= -2aQ_0(t)P_0(z) - 2bQ_2(t)P_2(z) \\ &= -a\log\left(\frac{1+t}{1-t}\right) - \frac{b}{4}(3z^2-1)\left((3t^2-1)\log\left(\frac{1+t}{1-t}\right) - 6t\right) \end{aligned} \tag{10.31}$$

This solution was first presented by Ferrari and Ibañez (1986, 1987a) in the case when $a = 1$ and $b = -2$. The solution with general parameters[1] was published by Griffiths (1987). The Ferrari–Ibañez (1986, 1987a) solution can be seen to be a modification of the Khan–Penrose solution, while the more general case is a generalization of the Szekeres solution (10.19) with $k_1 = k_2 = a$.

With the solution (10.31), the remaining equations (10.23) may be integrated to give

$$\begin{aligned} S = {} & \text{const} + (a+b)^2\log(t^2-z^2) - (a+b)^2\log(1-t^2) \\ & - 3(a+b)b\left[(1-z^2)t\log\left(\frac{1+t}{1-t}\right) - 2z^2\right] \\ & - \frac{9b^2}{4}(1-z^2)^2\left[\frac{1}{4}(1-t^2)(1-9t^2)\log^2\left(\frac{1+t}{1-t}\right)\right. \\ & \qquad\qquad \left. + t(7-9t^2)\log\left(\frac{1+t}{1-t}\right) + 9t^2 - 4\right] \\ & - \frac{9b^2}{4}(1-z^2)(1-t^2)\left[t^2\log^2\left(\frac{1+t}{1-t}\right) - 4t\log\left(\frac{1+t}{1-t}\right) + 4\right]. \end{aligned} \tag{10.32}$$

It may be noticed that S contains the term

$$(a+b)^2\log(t^2-z^2) = \tfrac{1}{2}(a+b)^2\log[16(\tfrac{1}{2}-f)(\tfrac{1}{2}+f)(\tfrac{1}{2}-g)(\tfrac{1}{2}+g)] \tag{10.33}$$

which includes the terms (7.10) that, with (7.8), are required to ensure that e^{-M} is continuous across the boundaries of region IV. If the leading terms in the power series for f and g take the form

$$f = \tfrac{1}{2} - (c_1u)^n + \dots, \qquad g = \tfrac{1}{2} - (c_2v)^n + \dots, \qquad n \geq 2, \tag{10.34}$$

then the required boundary conditions are satisfied if

$$(a+b)^2 = 2(1-1/n). \tag{10.35}$$

[1] This solution with $a + b = \pm 1$ has subsequently also been presented by Li (1989).

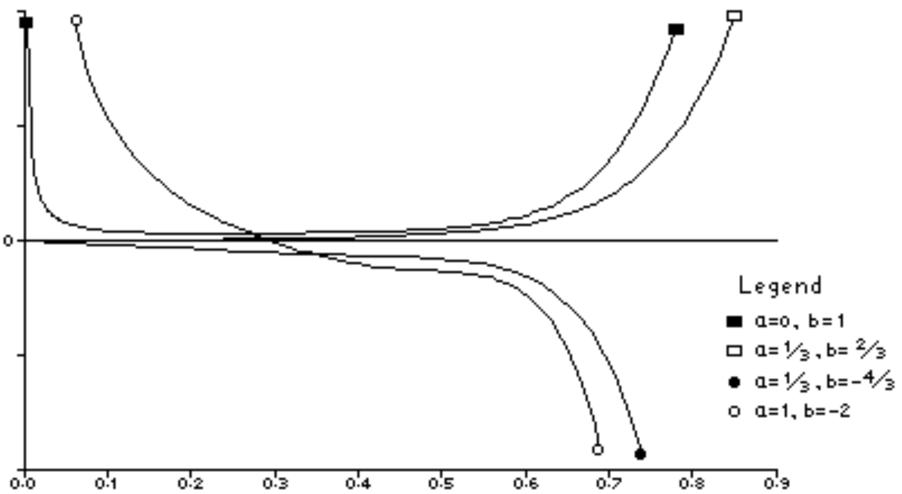


Figure 10.1 Wave profiles of approaching waves in the Ferrari–Ibañez–Griffiths solution with $n = 2$. These have been obtained from (10.6), but the scale factor contained in the bounded part of S has been removed. Profiles are shown for the special cases in which $b = 2/3$ or $-4/3$, when the wave front is continuous. The more general situation is represented by the cases in which b takes the values 1 and -2.

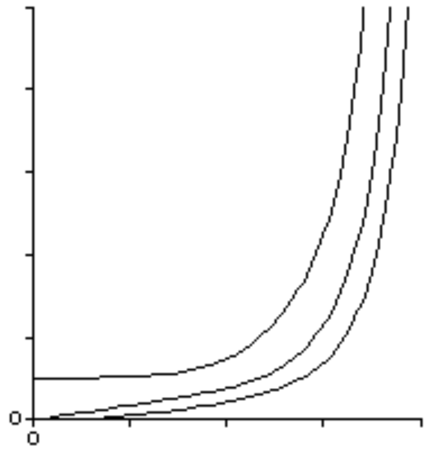
By extending this solution into regions II and III and putting $c_1 = c_2 = 1$, it can be seen from (10.6) that the approaching waves contain an impulsive wave component $\frac{1}{2}(2a - b)\delta(u)$ only if $n = 2$ and $a + b = \pm 1$. This case includes the Khan–Penrose solution for which $b = 0$ and the Ferrari–Ibañez (1987*a*) solution in which $a = 1$, $b = -2$. However, with $b \neq 0$, the impulsive wave is followed by another wave component. Some profiles for approaching waves of this type are illustrated in Figure 10.1.

It can also be shown using (10.6) that the wave front of the approaching wave in region II behaves as

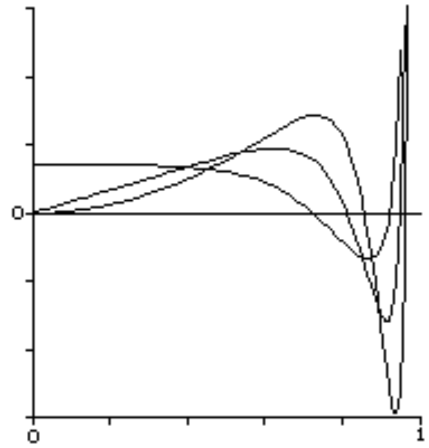
$$\begin{aligned}\Psi_4 = &\frac{n}{4}(2a-b)u^{(1/2-1/n)}\delta(u)\\ &+\frac{n^2}{32}(2a-b)\left((2a-b)^2-4\right)u^{n/2-2}\Theta(u)+O(u^2).\end{aligned}\tag{10.36}$$

Thus, unless $b = 2a$ or $2a - 2$, the wave front may have a distributional amplitude. It is unbounded if $2 \leq n < 4$ and has a step if $n = 4$. The wave front is smooth if $n > 6$. Scaled profiles for the approaching waves are illustrated in Figure 10.2 for some particular values of a and b with n equal to 4, 6 and 8.

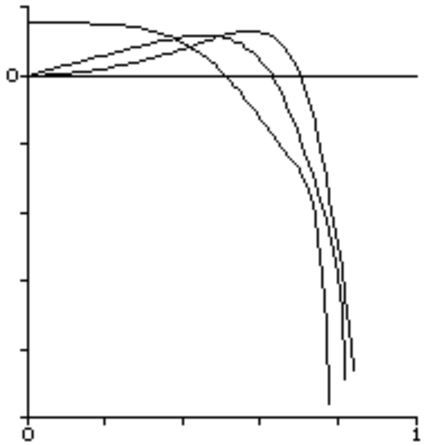
In all cases the approaching waves become unbounded as $f \to -\frac{1}{2}$ in region II, and as $g \to -\frac{1}{2}$ in region III.



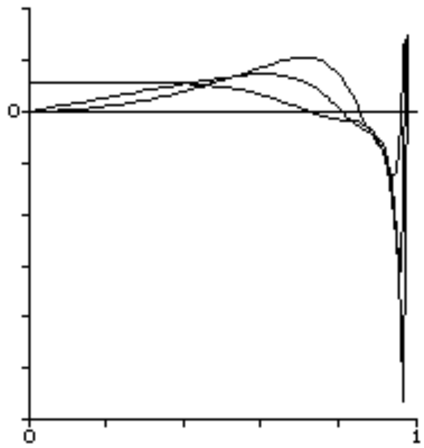
(a) $b = 1,\ a = \sqrt{2(1 - \frac{1}{n})} - 1.$



(b) $b = -\ 1,\ a = \sqrt{2(1 - \frac{1}{n})} + 1.$



(c) $b = -\ 2,\ a = 2 - \sqrt{2(1 - \frac{1}{n})}.$



(d) $b = -\ 2,\ a = 2 + \sqrt{2(1 - \frac{1}{n})}.$

Figure 10.2 Scaled wave profiles of approaching waves in the Ferrari–Ibañez–Griffiths solution for various values of a and b, and with n taking the values 4, 6 and 8.

It can also be shown that this family of solutions has the same singularity structure as the class of Szekeres solutions as described in Section 9.3 and to which it reduces when $b = 0$. In region IV, there is a scalar polynomial curvature singularity on the space-like surface $t = 1$ on which $\psi = \pi/2$ and $f + g = 0$. It may be noted that when $2 < n < 4$, the initial boundaries $u = 0$ and $v = 0$ of region IV contain a distribution-valued singularity. These initial boundaries are regular when $n \geq 4$. As in all other solutions, it may also be noted that there are non-scalar curvature

singularities in the initial regions II and III on the hypersurfaces on which $f=-\frac{1}{2}$ and $g=-\frac{1}{2}$ respectively.

10.4 The soliton solution of Ferrari and Ibañez

Ferrari and Ibañez (1987*b*) have also used a generating that is familiar in the study of solitons to obtain a solution of (10.2) in which

$$V=-2kQ_0(t)P_0(z)-\tfrac{1}{2}a\log(1-t^2)-\tfrac{1}{2}a\log(1-z^2) \tag{10.37}$$

where $k=1$ and a is an arbitrary constant.[2] This was initially obtained using the inverse scattering technique in which the initial 'seed' solution is taken to be the Kasner or Stoyanov solution given by (10.20) or (10.29).

This solution can immediately be seen to be a generalization of the Khan–Penrose solution (10.18) by the addition of the solution (10.20). It necessarily contains impulsive wave components.

It was pointed out by Griffiths (1987), however, that this solution may easily be generalized by treating the parameter k in (10.36) as another arbitrary constant. In this way, this family of solutions may also be considered to be a generalization of the Szekeres solutions with $k_1=k_2=k$. As in the previous example, this generalization then permits the approaching waves to have a continuous wave front for appropriate values of k and a. It is this more general class of solutions that is considered in this section.

It is possible immediately to integrate equations (10.23) to obtain

$$\begin{aligned}S=\ &\text{const}+k^2\log(t^2-z^2)-\tfrac{1}{4}a^2\log(1-z^2)\\&-\tfrac{1}{4}(a-2k)^2\log(1-t)-\tfrac{1}{4}(a+2k)^2\log(1+t).\end{aligned} \tag{10.38}$$

This contains the term $k^2\log(t^2-z^2)$ which, as in (10.33), is required to ensure that e^{-M} is continuous across the boundaries of region IV. If the leading terms in the power series for f and g take the form

$$f=\tfrac{1}{2}-(c_1u)^n+\ldots,\qquad g=\tfrac{1}{2}-(c_2v)^n+\ldots,\qquad n\geq 2, \tag{10.39}$$

then the required boundary conditions are satisfied if

$$k^2=2(1-1/n). \tag{10.40}$$

[2] As well as putting $k=1$, Ferrari and Ibañez find it convenient to use the parameters $s_1=\frac{1}{2}(1-a)$ and $s_2=\frac{1}{2}(1+a)$.

In this case it is remarkable that the junction conditions place no constraint on the parameter a. This feature is, in fact, related to a general result that will be described in Section 12.1.

When evaluating expressions for the Weyl tensor, it is convenient to use the coordinates ψ and λ defined by (10.10) (it may be noticed that these differ from those used by Ferrari and Ibañez). The scale-invariant components of the Weyl tensor are given by

$$\begin{aligned}
\Psi_0^\circ &= -\frac{g'^2}{\cos^2\psi\cos^2\lambda}\Bigg(\frac{2k(1-k^2)\cos^3\lambda}{\sin^3(\psi-\lambda)} + \frac{3k(ak-\sin\psi)\cos^2\lambda}{\sin^2(\psi-\lambda)} \\
&\qquad + \frac{3k(1-a^2)\cos\lambda}{\sin(\psi-\lambda)} - \frac{a}{2}(1-a^2)\Bigg) \\
\Psi_2^\circ &= \frac{f'g'}{\cos^2\psi\cos^2\lambda}\left(\frac{k(k-a\sin\psi)\cos^2\lambda}{\sin(\psi-\lambda)\sin(\psi+\lambda)} - \frac{1}{4}(1-a^2)\right) \\
\Psi_4^\circ &= -\frac{f'^2}{\cos^2\psi\cos^2\lambda}\Bigg(\frac{2k(1-k^2)\cos^3\lambda}{\sin^3(\psi+\lambda)} + \frac{3k(ak-\sin\psi)\cos^2\lambda}{\sin^2(\psi+\lambda)} \\
&\qquad + \frac{3k(1-a^2)\cos\lambda}{\sin(\psi+\lambda)} - \frac{a}{2}(1-a^2)\Bigg)
\end{aligned} \tag{10.41}$$

For the sake of later discussion, it is found to be appropriate to choose the scale functions A and B defined by (6.2) and (6.14) to be

$$\begin{aligned}
A &= -\frac{(f+g)^{1/4}}{f'}\frac{\sin^{k^2/2}(\psi+\lambda)}{\sin^{k^2/2}(\psi-\lambda)}\,e^{S/2} \\
&= -\frac{(\cos\psi\cos\lambda)^{(1-a^2)/4}\sin^{k^2}(\psi+\lambda)}{f'(1+\sin\psi)^{k(k+a)/2}(1-\sin\psi)^{k(k-a)/2}} \\
B &= -\frac{(f+g)^{1/4}}{g'}\frac{\sin^{k^2/2}(\psi-\lambda)}{\sin^{k^2/2}(\psi+\lambda)}\,e^{S/2} \\
&= -\frac{(\cos\psi\cos\lambda)^{(1-a^2)/4}\sin^{k^2}(\psi-\lambda)}{g'(1+\sin\psi)^{k(k+a)/2}(1-\sin\psi)^{k(k-a)/2}}.
\end{aligned} \tag{10.42}$$

The negative signs are due to the signs of f' and g'.

From (10.41) it can be seen that, in general, this solution has the same singularity structure as the Szekeres solution. There is a curvature singularity on the hypersurface given by $f+g=0$ or $\psi=\pi/2$.

There are, however, exceptional cases which occur when $a=\pm1$ and[3]

[3] Since V can always be replaced by $-V$, there is no loss of generality by assuming that k is always positive.

$k = 1$. In these cases

$$\Psi_0 = \mp\, 3\Psi_2 = \Psi_4 = \mp\, \frac{3}{(1 \pm \sin\psi)^3} \tag{10.43}$$

which satisfies the condition

$$\Psi_0\Psi_4 = 9\Psi_2^2 \tag{10.44}$$

which implies that the space-time is of algebraic type D (see Kramer *et al.* 1980, or Chandrasekhar and Xanthopoulos 1986*b*). These particular degenerate cases will be analysed in more detail in the next section.

When $n = 2$ and $k = 1$, the approaching waves contain an impulsive component. For $2 < n < 4$, $1 < k^2 < 3/2$, the null boundaries of region IV contain distribution-valued singularities. These boundaries are regular when $n \geq 4$, $\;3/2 \leq k^2 < 2$. The approaching waves have a step wavefront if $n = 4$, and the wavefront is continuous if $n > 6$.

Also, as in the Szekeres solutions, there are non-scalar curvature singularities in regions II and III when $f = -1/2$ and $g = -1/2$ respectively, for all values of a and all permissible values of k.

A generalization of the class of solutions described in this section has been obtained by Tsoubelis and Wang (1989). This has been obtained by putting

$$V = -\tfrac{1}{2} a \log(1 - t^2)(1 - z^2) - b_1 Q_0(t) P_0(z) - b_2 P_0(t) Q_0(z). \tag{10.45}$$

By a comparison with (10.37) and (10.19), this can be seen to be a generalization both of the above Ferrari–Ibañez solution and also of the Szekeres class of solutions described in Chapter 9. Its properties can reasonably be inferred from those of these two subclasses, as has been confirmed by Tsoubelis and Wang.

10.5 The degenerate Ferrari–Ibañez solutions

Consider now the special cases of the above Ferrari–Ibañez solution in which $k = 1$ and $a = \pm 1$, in which the Weyl tensor components are given by (10.43). The properties of these cases have been further described by Ferrari and Ibañez (1988).

In the case when $a = -1$, $k = 1$, there is again a curvature singularity when $\psi = \pi/2$. However, when $a = 1$ and $k = 1$, the space-time appears to be regular for all $0 \leq \psi \leq \pi/2$, and the singularity caused by the mutual focusing of the two waves appears to have been removed. It is, therefore, appropriate to consider this particular case in more detail.

Since $k=1$ in these cases, it is possible to scale the null coordinates such that

$$f=1-u^2\Theta(u), \qquad g=1-v^2\Theta(v). \tag{10.46}$$

When $a=1$, the metric functions in region IV are then given by

$$\begin{aligned} e^{-U} &= 1-u^2-v^2 \\ e^{V} &= \frac{1}{1-u^2-v^2}\left(\frac{1-u\sqrt{1-v^2}-v\sqrt{1-u^2}}{1+u\sqrt{1-v^2}+v\sqrt{1-u^2}}\right) \\ e^{-M} &= \frac{\left(1+u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)^2}{\sqrt{1-u^2}\sqrt{1-v^2}} \end{aligned} \tag{10.47}$$

and the Weyl tensor components are then given by

$$\begin{aligned} \Psi_0 &= \frac{1}{(1+u)^2\sqrt{1-u^2}}\,\delta(v) \;-\; \frac{3}{\left(1+u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)^3}\,\Theta(v) \\ \Psi_2 &= \frac{1}{\left(1+u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)^3}\,\Theta(u)\Theta(v) \\ \Psi_4 &= \frac{1}{(1+v)^2\sqrt{1-v^2}}\,\delta(u) \;-\; \frac{3}{\left(1+u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)^3}\,\Theta(u). \end{aligned} \tag{10.48}$$

As expected, these indicate that the approaching waves are impulses followed by continuous components. Of greater significance, however, is the fact that in this case there are two point singularities at the points $u=0, v=1$ and $u=1, v=0$. The existence of the singularities at these points implies that the lines $u=1$ and $v=1$ in regions II and III act as 'fold singularities' like those described in Section 8.2. These lines therefore form boundaries to the space-time in these regions.

It is convenient now to return to the time-like and space-like coordinates ψ and λ. In the degenerate case when $a=1$ the line element in the interaction region takes the form

$$\mathrm{d}s^2=\frac{(1+\sin\psi)^2}{2}(\mathrm{d}\psi^2-\mathrm{d}\lambda^2)-\left(\frac{1-\sin\psi}{1+\sin\psi}\right)\mathrm{d}x^2-\cos^2\lambda(1+\sin\psi)^2\mathrm{d}y^2. \tag{10.49}$$

To analyse this particular case further, consider the change of variables

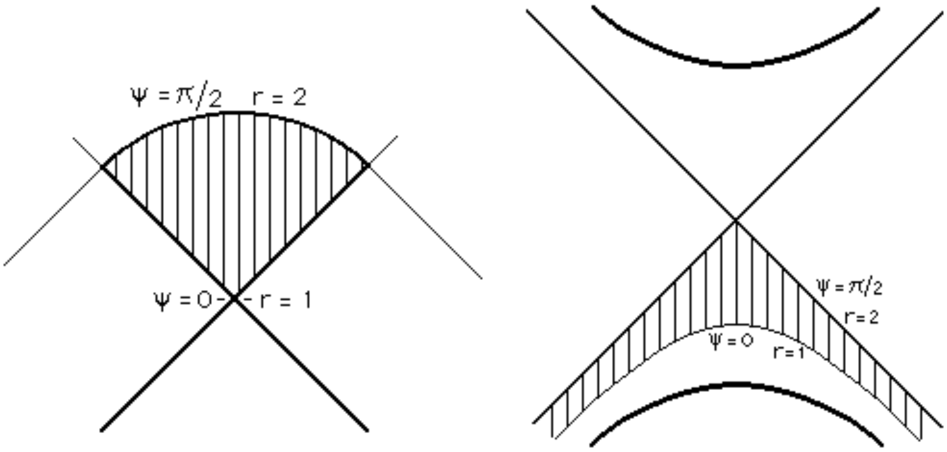
$$\begin{aligned} r &= 1+\sin\psi, \qquad & \tau &= \sqrt{2}x, \\ \theta &= \pi/2-\lambda, \qquad & \phi &= \sqrt{2}y. \end{aligned} \tag{10.50}$$

With this, the line element (10.49) takes the form

$$2\mathrm{d}s^2 = \left(1 - \frac{2}{r}\right)\mathrm{d}\tau^2 - \frac{1}{\left(1 - \frac{2}{r}\right)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \qquad (10.51)$$

which may immediately be recognized as the Schwarzschild metric with $m = 1$.

With $0 \leq \psi \leq \pi/2$, however, we have $1 \leq r \leq 2$ which is normally understood as the region inside the horizon. In addition, since ψ increases from zero in the interaction region, this region must correspond to the part of the Schwarzschild space-time indicated in Figure 10.3(b) which is inside the initial horizon. However, it may also be noticed that the coordinate ϕ which represents the axial coordinate in the Schwarzschild solution covers the entire range $-\infty < \phi < \infty$ in this case.



(a) $x =$ const, $y =$ const. (b) $\lambda =$ const, $y =$ const.

Figure 10.3 The interaction region of the degenerate solution viewed in two different coordinate planes. The plane (a) is the u,v or ψ,λ plane as in previous figures. Diagram (b) represents the ψ,x plane and is equivalent to the familiar Kruskal–Szekeres diagram for the Schwarzschild space-time.

From Figure 10.3, it appears that the approaching waves collide at the surface $\psi = 0$ and reach a horizon at $\psi = \pi/2$. There is, however, no *a priori* reason why the coordinate ψ should not be continued beyond $\pi/2$. If such a continuation is possible, this would indicate that the waves would continue through the horizon until they finally end in a curvature singularity at $\psi = 3\pi/2$.

The other degenerate case in which $a = -1$ can also be transformed to a Schwarzschild space-time but, in this case, the interaction region corresponds to the upper region inside the horizon and all trajectories end in the future singularity. This case, together with the above transformation, has also been described by Yurtsever (1988*a*).

It is instructive also to express the metric in Kruskal–Szekeres form. This can be achieved from (10.49) using the transformation

$$\begin{aligned} \tilde{u} &= \begin{cases} -\sqrt{1-\sin\psi}\, e^{(1+\sin\psi)/4}\, e^{\tau/4} & \text{for} \quad 0 \le \psi \le \pi/2 \\ +\sqrt{1-\sin\psi}\, e^{(1+\sin\psi)/4}\, e^{\tau/4} & \text{for} \quad \pi/2 < \psi \le 3\pi/2 \end{cases} \\ \tilde{v} &= \begin{cases} -\sqrt{1-\sin\psi}\, e^{(1+\sin\psi)/4}\, e^{-\tau/4} & \text{for} \quad 0 \le \psi \le \pi/2 \\ +\sqrt{1-\sin\psi}\, e^{(1+\sin\psi)/4}\, e^{-\tau/4} & \text{for} \quad \pi/2 < \psi \le 3\pi/2 \end{cases} \end{aligned} \tag{10.52}$$

where $\tilde{u}$ and $\tilde{v}$ are null coordinates in the ψ-x plane. With this, the line element becomes

$$2\mathrm{d}s^2 = \frac{16}{(1+\sin\psi)} e^{-(1+\sin\psi)/2} \mathrm{d}\tilde{u}\mathrm{d}\tilde{v} - (1+\sin\psi)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{10.53}$$

where ψ is given by

$$(1-\sin\psi)e^{\sqrt{1+\sin\psi}} = \tilde{u}\tilde{v}. \tag{10.54}$$

In these coordinates the structure of the solution in the vicinity of the horizon is clearly shown to be regular.

It is also convenient to consider this solution in terms of alternative null coordinates[4] u' and v' such that, in the interaction region

$$\begin{aligned} f &= \tfrac{1}{2} - \sin^2 au', & g &= \tfrac{1}{2} - \sin^2 bv' \\ \psi &= au' + bv', & \lambda &= au' - bv' \end{aligned} \tag{10.55}$$

where the new constants a and b are the strengths of the approaching gravitational shock waves. With these coordinates it can be seen that the Schwarzschild mass parameter, which is unity in the line elements (10.51) and (10.53), is here related to the amplitudes of the approaching waves by

$$m = \frac{1}{\sqrt{2ab}}. \tag{10.56}$$

[4] These coordinates were initially found useful by Bell and Szekeres (1974) and will be introduced here in Section 15.1. They have been used to analyse the degenerate Ferrari–Ibañez solution by Hayward (1989*a*).

It follows from this that stronger approaching gravitational waves would produce a shorter proper time between the collision and the subsequent horizon. They would also give rise to a smaller Schwarzschild mass parameter, and hence greater curvature on the horizon.

When trying to relate the global structure of this solution to part of the Schwarzschild space-time, it should be emphasised that for this class of colliding plane waves the coordinate ϕ is not periodic but covers the full range $-\infty < \phi < \infty$. Topological singularities occur on the lines $\theta = 0$ and $\theta = \pi$ on the horizon $\psi = \pi/2$ (or $r = 2$).

This global structure of the solution has been determined by Hayward (1989*a*). He has noted that the above solution also forms a covering space of part of the Schwarzschild white hole with a quasiregular covering space singularity along the polar axis around which the space-time is wound. He has also suggested an alternative extension of the Schwarzschild exterior inside the black hole.

In addition, he has suggested some possible extensions for the non-unique region beyond the horizon including one that is time symmetric. Refering to Figure 10.3(b), the opposing waves in this case collide on the hypersurface $\psi = 0$. The interaction region is then that part of the Schwarzschild space-time inside the initial horizon as indicated, which extends to the horizon as the focusing hypersurface. An extension through this surface may be taken to be the familiar two parts of the exterior Schwarzschild space-time. This can be continued to the subsequent horizon. A further extension through this horizon is then possible up to the next surface on which $\psi = 0$ or $r = 1$ at which the space-time again splits into two separating gravitational waves which are the time reverse of the initial approaching waves.

At the conclusion of this section, it may be noted that Ferrari and Ibañez (1988) have shown that the shear-free principal null congruences associated with this type D space-time do not focus on the horizon. However, these congruences are in different planes from the congruences on which the two wave components propagate, and for which the contraction and shear clearly become unbounded as $\psi \to \pi/2$.

10.6 An odd order solution

All of the solutions considered so far have involved only Legendre funtions of even order. A solution with odd order functions has been presented by Griffiths (1987)[5]. This has

$$V = aQ_1(t)P_1(z) \; = az\left(\frac{t}{2}\log\left(\frac{1+t}{1-t}\right) - 1\right). \tag{10.57}$$

[5] This solution with $a = 2$ has subsequently also been given by Li (1989).

With this function, equations (10.23) can immediately be integrated to give

$$\begin{aligned} S = &\ \text{const} + \tfrac{1}{4}a^2\log(t^2-z^2) - \tfrac{1}{4}a^2\log(1-z^2) \\ &- \tfrac{1}{4}a^2(1-z^2)\log\left(\frac{1+t}{1-t}\right)\left(\tfrac{1}{4}(1-t^2)\log\left(\frac{1+t}{1-t}\right)+t\right). \end{aligned} \tag{10.58}$$

As in (10.32–34), it can be clearly seen that this expression contains terms of the form (7.10). It may thus be concluded that, if the leading terms in the expansions for the functions f and g are given by (10.34), then the boundary conditions that are required for the solution to describe colliding plane waves are satisfied if

$$a^2/8 = 1 - 1/n. \tag{10.59}$$

As in previous examples, it can be shown that the approaching waves have an impulsive component if $n=2$. Also the wavefront is unbounded if $2<n<4$, has a step if $n=4$, is continuous but not smooth if $4<n\leq 6$, and is smooth if $n>6$. The expressions for the Weyl tensor components are rather complicated, and there is nothing particularly remarkable about this solution.

In the previous examples discussed in this chapter, the two waves approaching each other are identical. The component Ψ_0 in region III has an identical form to Ψ_4 in region II but with the null coordinate v replacing u. Similarly, the expressions for V in the two regions can be related by the same replacement. In this case, however, since odd functions are being used, a change of sign is also included. Thus, $V(u)$ and $\Psi_4(u)$ in region II are replaced by $-V(v)$ and $-\Psi_0(v)$ in region III. The approaching waves still have colinear polarization but, in this case, their amplitudes are opposite rather than the same.

10.7 The second Yurtsever and the Feinstein–Ibañez solutions

For colliding plane wave solutions, we are considering space-times with two space-like Killing vectors. Such space-times have also been considered in the context of cosmology, where certain vacuum inhomogeneous cosmologies satisfy the same field equations as the interaction region for colliding colinear gravitational waves (Gowdy 1971), although different boundary conditions are appropriate. In fact the vacuum Gowdy cosmologies can be considered to represent closed universes built from opposing plane gravitational waves. The similarity between these solutions

has been noted by Feinstein and Ibañez (1989), who have used known cosmological solutions to obtain a new class of colliding plane wave solutions. In this section, the Feinstein–Ibañez solutions are presented. The electromagnetic Gowdy cosmologies will be described in Section 17.3.

It is appropriate at this point to consider the alternative system of coordinates defined by

$$\begin{aligned}\tilde t &= f+g=\sqrt{1-t^2}\sqrt{1-z^2}\\ \tilde z &= f-g=-tz.\end{aligned} \tag{10.60}$$

It may be noted that the new coordinate $\tilde t$ is a decreasing, or past pointing, time-like coordinate, and that the singularity in region IV occurs when $\tilde t=0$. These coordinates have also been used by Yurtsever (1988c)[6].

In this case, the line element (10.1) for colliding waves with aligned constant polarization can be rewritten in the diagonal form

$$\mathrm{d}s^2=\frac{e^{-S}}{2\sqrt{\tilde t}}\left(\mathrm{d}\tilde t^2-\mathrm{d}\tilde z^2\right)-\tilde t\left(e^V\mathrm{d}x^2+e^{-V}\mathrm{d}y^2\right) \tag{10.61}$$

and the main vacuum field equation (9.3), (10.2) or (10.13) becomes

$$\ddot V+\frac{1}{\tilde t}\dot V-V''=0 \tag{10.62}$$

and the subsidiary equations (10.3) become

$$\dot S=\tfrac12\tilde t\left(\dot V^2+V'^2\right),\qquad S'=-\tilde t\dot V V' \tag{10.63}$$

where the dot and prime denote derivatives with respect to $\tilde t$ and $\tilde z$ respectively.

The above equations (10.62,63) are exactly those for the vacuum Gowdy cosmologies that have been considered by many authors.[7] The general solution of (10.62) can be expressed as a line integral. However, for the class of Gowdy cosmologies, it is found to be convenient to consider the class of solutions given by

$$\begin{aligned}V=&-a\log\tilde t+L_1\{A_\omega\cos[\omega(\tilde z+\alpha_\omega)]J_0(\omega\tilde t)\}\\ &+L_2\{B_\omega\cos[\omega(\tilde z+\beta_\omega)]Y_0(\omega\tilde t)\}+\sum_i d_i\cosh^{-1}\left(\frac{\tilde z+c_i}{\tilde t}\right)\end{aligned} \tag{10.64}$$

---

[6] In the notation used by Yurtsever $\tilde t$ and $\tilde z$ are replaced by α and β respectively.

[7] See for example Centrella and Matzner (1979), Moncrief (1981), Carmeli, Charach and Malin (1981), Carr and Verdaguer (1983), Ibañez and Verdaguer (1983), Adams *et al.* (1982), Feinstein and Charach (1986), and Feinstein (1987).

where $L_1\{\ \}$ and $L_2\{\ \}$ are arbitrary linear combinations of the terms in curly brackets, including the Fourier–Bessel integrals of the form

$$\begin{aligned} &\int_0^\infty B_\omega \cos[\omega(\tilde{z}+\beta_\omega)] Y_0(\omega\tilde{t})\mathrm{d}\omega \\ &\int_0^\infty A_\omega \cos[\omega(\tilde{z}+\alpha_\omega)] J_0(\omega\tilde{t})\mathrm{d}\omega. \end{aligned} \tag{10.65}$$

$J_0(\omega\tilde{t})$ and $Y_0(\omega\tilde{t})$ are Bessel functions of the first and second kinds of zero order, and A_ω, B_ω, α_ω and β_ω are sets of arbitrary constants.

The first term in (10.64) is clearly identical to the solution (10.20). It is the term that, on its own, gives rise to the Kasner solutions. The set of terms included in L_1 have regular behaviour as $\tilde{t} \to 0$. It may be noted that this property is also shared by the combination of terms $\sum a_n P_n(t)P_n(z)$ that are contained in (10.16). The set of terms included in L_2, however, are badly behaved as $\tilde{t} \to 0$. These are the terms that are considered to induce chaotic behaviour in these cosmological models. The last term in (10.64) is identical to (10.31). This contains the so-called gravitational soliton components.

Yurtsever (1988*c*) has considered the asymptotic structure of those solutions which only contain the combinations $L_1\{\ \}$ and $L_2\{\ \}$, i.e. in which $a = d_i = 0$. He has shown that these solutions are asymptotic to the inhomogeneous Kasner solutions as the singularity $\tilde{t} = 0$ is approached. He has also given explicit expressions which relate the asymptotic Kasner exponents along the singularity to the initial data specified along the wave fronts of the incoming colliding plane waves. It follows from this analysis that the focusing hypersurface $\tilde{t} = 0$ is a curvature singularity except in the special case in which one of the Kasner exponents is zero. This special case of the degenerate Kasner solution is flat and, in this case, the focusing hypersurface is a Killing–Cauchy horizon across which space-time can be extended. It is reasonable to conclude from this that, although there are colliding plane wave space-times which contain a Killing–Cauchy horizon rather than a space-like curvature singularity, these space-times are unstable against small perturbations of the initial data and that 'generic' initial data always produce space-like curvature singularities.

Feinstein and Ibañez (1989) have considered the family of solutions for which the combination L_2 is zero. They have shown that, for this case, a curvature singularity does not develop in region IV as $\tilde{t} \to 0$ provided that

$$\sum d_i = a \pm 1. \tag{10.66}$$

They have also shown that, in this case, the necessary boundary conditions (7.15) are satisfied provided there are at least two solitonic terms

with constants satisfying

$$\begin{aligned} c_1 &= -1, \qquad & {d_1}^2 &= 2(1-1/n_1) \\ c_2 &= 1, \qquad & {d_2}^2 &= 2(1-1/n_2) \end{aligned} \tag{10.67}$$

using the notation of (7.11). (The unfortunate repeated use of constants c_1 and c_2 in this equation should not cause confusion.) It may thus be noted that these two solitonic terms provide the conditions for continuity on the two different null boundaries of region IV. For continuity it is also required that $n_1 \geq 2$ and $n_2 \geq 2$, so d_1 and d_2 are constrained to the range

$$1 \leq |d_1|, |d_2| \leq \sqrt{2}. \tag{10.68}$$

It follows from this and (10.66) that $a \neq 0$, and so the first term in (10.64) must necessarily be included in these solutions.

It may also be noted that the solutions given in the notation of previous sections by

$$\begin{aligned} V = &-\tfrac{1}{2}a\log(1-t^2) - \tfrac{1}{2}a\log(1-z^2) \\ &+ \sum_n a_n P_n(t)P_n(z) + \sum_i d_i \cosh^{-1}\left(\frac{c_i - tz}{\sqrt{1-t^2}\sqrt{1-z^2}}\right) \end{aligned} \tag{10.69}$$

where c_1 and c_2 are given by (10.67), similarly do not contain curvature singularities on the hypersurface $f+g=0$ provided the constants d_i and a are constrained by (10.66) and (10.68).

It is also of interest to note that the two solitonic terms that provide the continuity across the boundaries of the interaction region are in fact identical to the two separate terms in the Szekeres solution (9.4). This may be observed by noting that

$$\begin{aligned} \cosh^{-1}\left(\frac{1+tz}{\sqrt{1-t^2}\sqrt{1-z^2}}\right) &= \cosh^{-1}\left(\frac{1-f+g}{f+g}\right) \\ &= 2\tanh^{-1}\left(\frac{\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}}\right) \\ &= -\log\left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f}}\right) \\ &= -\tfrac{1}{2}\log\left(\frac{1+t}{1-t}\right) - \tfrac{1}{2}\log\left(\frac{1+z}{1-z}\right) \\ &= -Q_0(t)P_0(z) - P_0(t)Q_0(z) \end{aligned} \tag{10.70}$$

and similarly

$$\cosh^{-1}\left(\frac{1-tz}{\sqrt{1-t^2}\sqrt{1-z^2}}\right) = -Q_0(t)P_0(z) + P_0(t)Q_0(z). \qquad (10.71)$$

It follows from this that even the Szekeres solution, which is everywhere at least C^2, can be adapted by the inclusion of a suitable multiple of the term (10.20) to provide a solution without a curvature singularity in the interaction region. Thus, the occurrence of a quasiregular singularity in region IV can have nothing to do with the relaxation of the continuity conditions across the boundaries.

The class of solutions included here contain an arbitrary number of parameters. It can be shown that they are algebraically general in the interaction region. They contain the usual coordinate singularity on the hypersurface $f+g=0$ but, for this class, this is not a curvature singularity. Feinstein and Ibañez have shown that the solution is extendable across this surface. However, the extension is not unique.

10.8 The first Yurtsever solutions

For colliding plane waves with aligned linear polarization, it has been seen that the main field equation (10.2) is linear. Various classes of solutions have already been obtained by separating the variables of this equation in a number of different ways. A further class of solutions in which the main equation is separated in yet another way has been given by Yurtsever (1988*a*). These solutions were originally obtained indirectly by first considering the Weyl solutions for stationary axisymmetric space-times. However, they will be presented here in a more general way that is also more appropriate in considering colliding wave solutions.

It is appropriate here to start with the main equation in the form (10.62) using the variables $\tilde t$ and $\tilde z$ defined by (10.60), and then transforming it by putting

$$\tilde t = \nu\sinh\eta, \qquad \tilde z = \nu\cosh\eta \qquad (10.72)$$

where the parameters ν and η are not necessarily real. Indeed, real values of these variables only cover part of the interaction region as will be clarified later. With these parameters, the main equation (10.62) becomes

$$V_{\nu\nu} + \frac{2}{\nu}V_\nu - \frac{1}{\nu^2}\coth\eta\, V_\eta - \frac{1}{\nu^2}V_{\eta\eta} = 0 \qquad (10.73)$$

and this has a series of separable solutions of the form

$$V = \sum_n \left(c_n\nu^n + d_n\nu^{-n-1}\right)\left(a_nP_n(\cosh\eta) + b_nQ_n(\cosh\eta)\right) \qquad (10.74)$$

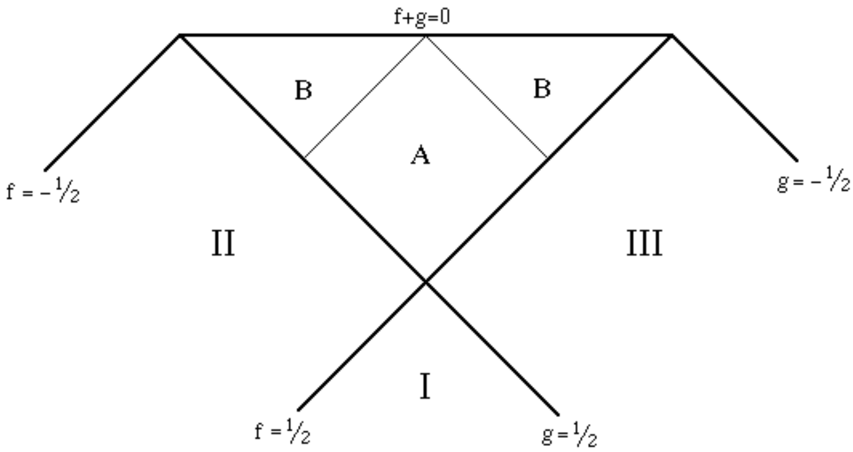


Figure 10.4 In the interaction region IV for the Yurtsever solutions, the variables ν and $\cosh\eta$ are imaginary in the initial region marked A that immediately follows the collision. This region is followed by the regions marked B in which these variables are real.

where $P_n(\cosh\eta)$ and $Q_n(\cosh\eta)$ are Legendre functions of the first and second kinds.

We must now reconsider the parameters of these solutions. As defined by (10.72), ν and $\cosh\eta$ are imaginary when the product fg is positive. Since both f and g decrease from $\frac{1}{2}$ towards $-\frac{1}{2}$ in the interaction region, it is appropriate to divide this region into the subregions A and B in which the product fg is positive and negative respectively as indicated in Figure 10.4. Explicitly we may put

$$\begin{aligned} \text{In A}: \qquad & \nu = 2i\sqrt{fg}, \qquad \cosh\eta = -\frac{i(f-g)}{2\sqrt{fg}} \\ \text{In B}: \qquad & \nu = 2\sqrt{-fg}, \qquad \cosh\eta = \frac{(f-g)}{2\sqrt{-fg}}. \end{aligned} \tag{10.75}$$

It is clear that ν is zero midway along the two boundaries between region IV and regions II and III. The inverse functions of ν that appear in (10.74) must therefore be excluded for colliding plane wave solutions since they must be regular along these boundaries. It is therefore appropriate to express general solutions in the form

$$\begin{aligned} \text{In A}: \qquad & V = \sum_n i^n (fg)^{n/2} \left(a_n P_n \left(\frac{i(g-f)}{2\sqrt{fg}} \right) + b_n Q_n \left(\frac{i(g-f)}{2\sqrt{fg}} \right) \right) \\ \text{In B}: \qquad & V = \sum_n (-fg)^{n/2} \left(a_n P_n \left(\frac{(f-g)}{2\sqrt{-fg}} \right) + b_n Q_n \left(\frac{(f-g)}{2\sqrt{-fg}} \right) \right). \end{aligned} \tag{10.76}$$

This solution will from now on simply be quoted in the second form since either equation formally includes the other.

It may be noted that these solutions are continuous on the boundaries between the subregions A and B. Also, the terms involving the Legendre functions of the first kind are regular on the focusing hypersurface $f+g=0$, while those involving the Legendre functions of the second kind are singular on this surface.

To be precise, in the original class of solutions given by Yurtsever (1988*a*), V is taken in the form

$$\begin{aligned} V = & -2Q_0(t)P_0(z) + \tfrac{1}{2}\log(1-t^2) + \tfrac{1}{2}\log(1-z^2) \\ & + \sum_n a_n(-fg)^{n/2} P_n\left(\frac{(f-g)}{2\sqrt{-fg}}\right) \end{aligned} \tag{10.77}$$

which can be seen to be generalizations of the degenerate Ferrari–Ibañez solution described in Section 10.5. They are therefore distortions of the Schwarzschild black hole solution in the interaction region, and correspond to interior Weyl solutions for static axisymmetric space-times. In terms of colliding plane waves, these solutions all involve approaching waves with initial impulsive components. Some particular examples have been described explicitly by Yurtsever (1988*a*).

Clearly these solutions can easily be generalised by modifying the coefficients of the first terms of (10.77). By choosing these appropriately, it is possible to remove the impulsive components from the approaching waves.

10.9 Further explicit solutions

To obtain solutions describing the collision and interaction of plane waves with aligned linear polarization it is necessary first to solve the main field equation, which may be expressed in terms of different coordinates in any of the forms (9.3), (10.2), (10.13), (10.62) or (10.73).

Any number of further explicit solutions may easily be generated using the methods described in the previous sections of this chapter with different combinations of particular solutions of the main equation (9.3), (10.2), (10.13), (10.62) or (10.73). It is appropriate here simply to list the various possibilities.

First there are the solutions (10.16) which have been obtained by separating the variables in (10.13). These involve Legendre functions of the first and second kinds

$$V_1 = \sum_n \big[a_nP_n(t) + q_nQ_n(t)\big]P_n(z) + \sum_n \big[p_nP_n(t) + b_nQ_n(t)\big]Q_n(z) \tag{10.78}$$

where a_n, q_n, p_n and b_n are series of arbitrary constants. It may be noted, of course, that non-integer values of n may also be included.

There are also the particular solutions (10.20), (10.8) and (10.22) that can be re-expressed in the form

$$\begin{aligned} V_2 = -a\log(f+g) + \sum_i \frac{A_i}{\sqrt{\sigma_i - f}\sqrt{\sigma_i + g}} \\ + \sum_i d_i \cosh^{-1}\left(\frac{c_i + f - g}{f+g}\right) \end{aligned} \tag{10.79}$$

where a, A_i, σ_i, c_i and d_i are arbitrary constants.

Then there are the solutions of (10.62) that are contained in (10.64) and have previously been included in Gowdy cosmologies, namely

$$V_3 = \sum_\omega A_\omega \cos[\omega(\tilde z + \alpha_\omega)] J_0(\omega\tilde t) + \sum_\omega B_\omega \cos[\omega(\tilde z + \beta_\omega)] Y_0(\omega\tilde t) \tag{10.80}$$

where $J_0(\omega\tilde t)$ and $Y_0(\omega\tilde t)$ are Bessel functions of the first and second kinds of zero order, $\tilde t = f + g$, $\tilde z = f - g$, and A_ω, B_ω, α_ω and β_ω are sets of arbitrary constants.

It may be noted that the solutions (10.80) have been obtained by separating the variables in (10.62), taking only solutions that are periodic in $\tilde z$. This constraint is not required for colliding plane waves and an additional set of aperiodic solutions is given by

$$V_4 = \sum_\omega A_\omega \cosh[\omega(\tilde z + \alpha_\omega)] I_0(\omega\tilde t) + \sum_\omega B_\omega \cosh[\omega(\tilde z + \beta_\omega)] K_0(\omega\tilde t) \tag{10.81}$$

where $I_0(\omega\tilde t)$ and $K_0(\omega\tilde t)$ are modified Bessel functions of the first and second kinds of zero order.

Finally, there are the separable solutions of (10.73) which are non-singular on the initial boundaries, namely

$$V_5 = \sum_n (-fg)^{n/2} \left(a_n P_n\left(\frac{(f-g)}{2\sqrt{-fg}}\right) + b_n Q_n\left(\frac{(f-g)}{2\sqrt{-fg}}\right)\right) \tag{10.82}$$

which again involve Legendre functions of the first and second kind.

When considered as infinite series, these different forms may simply be regarded as different representations of the same class of solutions. However, when looking for particular explicit solutions it is necessary to consider only a few terms. In this way, further exact solutions for colliding plane waves may be obtained by combining particular components from

any of the series V_1, V_2, V_3, V_4 and V_5 and by choosing the arbitrary constants such that

$$\lim_{g\to 1/2}\left[(\tfrac{1}{2}-g)V_g^2\right]=2k_2, \qquad \lim_{f\to 1/2}\left[(\tfrac{1}{2}-f)V_f^2\right]=2k_1 \tag{10.83}$$

where k_1 and k_2 are constrained to the range (7.13). If this condition is satisfied, then it remains only to integrate the subsidiary equations in the form (10.3) or (10.63) to obtain the remaining metric function S or M.

11

ERNST'S EQUATION FOR COLLIDING GRAVITATIONAL WAVES

The solutions being considered here for colliding plane waves all have a pair of commuting Killing vectors that are assumed to exist globally. It may therefore be expected that the solutions obtained may be related to known cylindrically symmetric solutions, or to stationary axisymmetric solutions, which similarly have a pair of Killing vectors. Such a relation was first pointed out by Kinnersley (1975), and by Fisher (1980). The exact relation between these solutions has been established more recently by Chandrasekhar and Ferrari (1984), and Chandrasekhar and Xanthopoulos (1985*a*), and exploited by these authors and their colleagues.

In this chapter we will present an analysis of the colliding wave problem using a method that has become familiar in the study of stationary axisymmetric space-times. In this case the field is described in terms of a complex potential function that is referred to as the Ernst potential (see Ernst 1968*a*). This approach leads directly to various methods for generating further exact solutions.

11.1 A derivation of the Ernst equation

First it may be recalled that, of the vacuum field equations (6.22*a*–*f*) considered in previous chapters, (6.22*a*) may immediately be integrated to give

$$e^{-U} = f(u) + g(v), \tag{11.1}$$

and (6.22d, e) are integrability conditions for the remaining equations. Attention is thus focused on these main equations for the metric functions $V(u,v)$ and $W(u,v)$.

It is now appropriate to consider a different combination of these functions, by putting

$$\chi(u,v) = e^{-V}\mathrm{sech}W, \qquad \omega(u,v) = e^{-V}\tanh W, \tag{11.2}$$

or, inversely

$$e^{V} = (\omega^2+\chi^2)^{-1/2}, \qquad \sinh W = \omega/\chi. \tag{11.3}$$

This modifies the form of the line element (6.20), which now becomes

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - e^{-U}\left(\chi\,\mathrm{d}y^2 + \chi^{-1}(\mathrm{d}x - \omega\mathrm{d}y)^2\right). \tag{11.4}$$

It is also convenient to introduce the complex function

$$Z = \chi + i\omega. \tag{11.5}$$

With this, the line element (11.4) can be written in the form

$$\mathrm{d}s^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - \frac{2e^{-U}}{(Z+\bar{Z})}(\mathrm{d}x + iZ\,\mathrm{d}y)(\mathrm{d}x - i\bar{Z}\,\mathrm{d}y) \tag{11.6}$$

and the two main equations (6.22d, e) can be written as the single complex equation

$$(Z+\bar{Z})(2Z_{uv} - U_uZ_v - U_vZ_u) = 4Z_uZ_v. \tag{11.7}$$

It may be seen that this is in fact Ernst's equation, which can be written in the coordinate-invariant form

$$(Z+\bar{Z})\nabla^2 Z = 2(\nabla Z)^2 \tag{11.8}$$

where $(\nabla\phi)^2 = g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ is the square of the gradient of an arbitrary scalar field ϕ which, in this case, is a function of the two (null) coordinates only. Similarly, ∇^2 is the 3+1-dimensional Laplacian operator (or the generalized d'Alembertian) given by

$$\nabla^2\phi = (g^{\mu\nu}\phi_{,\nu})_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\mu\nu}\phi_{,\nu}\right)_{,\mu}. \tag{11.9}$$

Using the above notation, the non-zero components of the Weyl tensor given by (6.23) can be written as

$$\begin{aligned}
\Psi_0^{\circ} &= \frac{|Z|}{Z(Z+\bar{Z})^2}\left[(Z+\bar{Z})(Z_{vv} - U_vZ_v + M_vZ_v) - 2Z_v^2\right]\\
\Psi_2^{\circ} &= -\frac{U_uU_v}{4} - \frac{\bar{Z}_uZ_v}{(Z+\bar{Z})^2}\\
\Psi_4^{\circ} &= \frac{|Z|}{Z(Z+\bar{Z})^2}\left[(Z+\bar{Z})(Z_{uu} - U_uZ_u + M_uZ_u) - 2Z_u^2\right].
\end{aligned} \tag{11.10}$$

When considering Ernst's equation, it is frequently found to be useful also to introduce an associated function E, defined by

$$Z = \frac{1+E}{1-E}, \qquad \text{or} \qquad E = \frac{Z-1}{Z+1}. \tag{11.11}$$

With this, the line element (11.4) or (11.6) can be written in the alternative form

$$\begin{aligned} ds^2 &= 2e^{-M}\mathrm{d}u\mathrm{d}v \\ &\quad - \frac{e^{-U}}{1-E\bar{E}}\big[(1-E)\mathrm{d}x + i(1+E)\mathrm{d}y\big]\big[(1-\bar{E})\mathrm{d}x - i(1+\bar{E})\mathrm{d}y\big] \end{aligned} \tag{11.12}$$

and the main equations ($6.22d, e$), or alternatively (11.7), become

$$(1-E\bar{E})(2E_{uv} - U_uE_v - U_vE_u) = -4\bar{E}E_uE_v. \tag{11.13}$$

This is the alternative form of the Ernst equation and may be rewritten in the coordinate-invariant form

$$(1-E\bar{E})\nabla^2E = -2\bar{E}(\nabla E)^2. \tag{11.14}$$

Using this function, the non-zero components of the Weyl tensor (11.10) can be written as

$$\begin{aligned} \Psi_0^{\circ} &= \sqrt{\frac{1-\bar{E}^2}{1-E^2}}\frac{1}{(1-E\bar{E})^2}\left((1-E\bar{E})(E_{vv} - U_vE_v + M_vE_v) + 2\bar{E}E_v^2\right) \\ \Psi_2^{\circ} &= -\frac{U_uU_v}{4} - \frac{\bar{E}_uE_v}{(1-E\bar{E})} \\ \Psi_4^{\circ} &= \sqrt{\frac{1-\bar{E}^2}{1-E^2}}\frac{1}{(1-E\bar{E})^2}\left((1-E\bar{E})(E_{uu} - U_uE_u + M_uE_u) + 2\bar{E}E_u^2\right). \end{aligned} \tag{11.15}$$

The problem now involves finding appropriate solutions of (11.7) or (11.13). These equations, however, contain the derivatives of U, and hence they depend on the arbitrary functions $f(u)$ and $g(v)$ that are specified by the incoming waves. This apparently explicit dependence on initial conditions may in fact be removed by adapting the coordinate system of Chandrasekhar and Ferrari (1984). Here we again use (10.9–12) and put

$$f(u) = \tfrac{1}{2}\cos(\psi+\lambda), \qquad g(v) = \tfrac{1}{2}\cos(\psi-\lambda) \tag{11.16}$$

where ψ and λ are considered as time-like and space-like coordinates, which may then be rescaled by putting

$$t = \sin\psi, \qquad z = \sin\lambda. \tag{11.17}$$

In this coordinate system, equation (11.7) takes the more familiar explicit form of Ernst's equation:

$$
\begin{aligned}
(Z+\bar{Z})\Big(\big((1-t^2)Z_t\big)_{,t} - \big((1-z^2)Z_z\big)_{,z}\Big)& \\
= 2\left((1-t^2)Z_t^2 - (1-z^2)Z_z^2\right)&
\end{aligned}
\tag{11.18}
$$

and (11.13) similarly becomes

$$
\begin{aligned}
(1-E\bar{E})\Big(\big((1-t^2)E_t\big)_{,t} - \big((1-z^2)E_z\big)_{,z}\Big)& \\
= -2\bar{E}\left((1-t^2)E_t^2 - (1-z^2)E_z^2\right).&
\end{aligned}
\tag{11.19}
$$

The intermediate steps in the derivation of these equations may be deduced from (16.6) and (16.11). See also (12.30–31).

It may be noticed that, in this case, the solution of these equations immediately determines some of the metric functions. This is in marked contrast to their application in stationary axisymmetric space-times, where the Ernst equation only determines potentials for the fields.

The original metric functions, as considered in previous chapters, are now given by

$$
\begin{aligned}
e^{2V} &= (Z\bar{Z})^{-1} = \frac{(1-E)(1-\bar{E})}{(1+E)(1+\bar{E})} \\
\sinh W &= -i\frac{(Z-\bar{Z})}{(Z+\bar{Z})} = -i\frac{(E-\bar{E})}{(1-E\bar{E})}.
\end{aligned}
\tag{11.20}
$$

11.2 Boundary conditions

When looking for solutions of the Ernst equation for stationary axisymmetric space-times, it is appropriate to require that solutions be asymptotically flat. However, for colliding plane waves very different boundary conditions apply.

For colliding plane waves it is necessary to choose Z or E, and hence V and W, such that the solution of $(6.22b, c, f)$ for M is continuous across the boundaries of region IV. For vacuum solutions, it is appropriate to use (7.8) and equations (7.9) may then be written in the form

$$
\begin{aligned}
S_f &= -2(f+g)(1-E\bar{E})^{-2}E_f\bar{E}_f \\
S_g &= -2(f+g)(1-E\bar{E})^{-2}E_g\bar{E}_g.
\end{aligned}
\tag{11.21}
$$

To ensure that the boundary conditions are satisfied, it is then essential that the solution of these equations should include the necessary components (7.10).

In this approach, solutions in the interaction region for $Z(t,z)$ or $E(t,z)$ are related to the functions $V(f,g)$ and $W(f,g)$, and hence S may be obtained as a function of f and g. The boundary conditions described in Chapter 7 may then be considered to place restrictions on the structure of these functions, which characterize the approaching waves.

In practice, these boundary conditions are difficult to apply, basically because the condition that M be continuous is only indirectly applied to the functions V and W, or Z, or E, that feature in the main equations. It is therefore often convenient to apply the boundary condition in the form (7.15) or (7.16). Writing $Z = Z(u,v)$, this becomes

$$\begin{aligned} \lim_{\substack{v\to 0\\ u\to 0}} \left[\frac{2}{n_2(n_2-1)v^{n_2-2}} \frac{Z_v\bar{Z}_v}{(Z+\bar{Z})^2} \right] &= c_2^2 \\ \lim_{\substack{u\to 0\\ v\to 0}} \left[\frac{2}{n_1(n_1-1)u^{n_1-2}} \frac{Z_u\bar{Z}_u}{(Z+\bar{Z})^2} \right] &= c_1^2. \end{aligned} \tag{11.22}$$

This form is particularly convenient when $n_1 = n_2 = 2$, which occurs when impulsive waves are present. The boundary conditions in this limited case only, have been discussed by Ernst, García-Díaz and Hauser (1987*b*).[1]

In terms of the functions f and g, the conditions (11.22) become

$$\begin{aligned} \lim_{\substack{g\to 1/2\\ f\to 1/2}} \left[(\tfrac{1}{2}-g) \frac{Z_g\bar{Z}_g}{(Z+\bar{Z})^2} \right] &= \frac{k_2}{2} \\ \lim_{\substack{f\to 1/2\\ g\to 1/2}} \left[(\tfrac{1}{2}-f) \frac{Z_f\bar{Z}_f}{(Z+\bar{Z})^2} \right] &= \frac{k_1}{2} \end{aligned} \tag{11.23}$$

where k_1 and k_2 must satisfy the inequalities (7.13).

Alternatively, writing $E = E(u,v)$, the boundary conditions require that

$$\begin{aligned} \lim_{\substack{v\to 0\\ u\to 0}} \left[\frac{2}{n_2(n_2-1)v^{n_2-2}} \frac{E_v\bar{E}_v}{(1-E\bar{E})^2} \right] &= c_2^2 \\ \lim_{\substack{u\to 0\\ v\to 0}} \left[\frac{2}{n_1(n_1-1)u^{n_1-2}} \frac{E_u\bar{E}_u}{(1-E\bar{E})^2} \right] &= c_1^2 \end{aligned} \tag{11.24}$$

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[1] For an alternative approach to the formulation of the boundary conditions see Hauser and Ernst (1989*a*).

which, in terms of the functions f and g become

$$\begin{aligned}
\lim_{\substack{g\to 1/2\\ f\to 1/2}} \left[(\tfrac{1}{2} - g) \frac{E_g \bar{E}_g}{(1 - E\bar{E})^2} \right] &= \frac{k_2}{2} \\
\lim_{\substack{f\to 1/2\\ g\to 1/2}} \left[(\tfrac{1}{2} - f) \frac{E_f \bar{E}_f}{(1 - E\bar{E})^2} \right] &= \frac{k_1}{2}.
\end{aligned} \tag{11.25}$$

Finally, it is convenient to express the boundary conditions in terms of the variables t and z. These become, for $Z = Z(t, z)$,

$$\begin{aligned}
\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(Z_t + Z_z)(\bar{Z}_t + \bar{Z}_z)}{(Z + \bar{Z})^2} \right] &= 2k_1 \\
\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(Z_t - Z_z)(\bar{Z}_t - \bar{Z}_z)}{(Z + \bar{Z})^2} \right] &= 2k_2
\end{aligned} \tag{11.26}$$

and for $E = E(t, z)$,

$$\begin{aligned}
\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(E_t + E_z)(\bar{E}_t + \bar{E}_z)}{(1 - E\bar{E})^2} \right] &= 2k_1 \\
\lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(E_t - E_z)(\bar{E}_t - \bar{E}_z)}{(1 - E\bar{E})^2} \right] &= 2k_2
\end{aligned} \tag{11.27}$$

where k_1 and k_2 satisfy (7.13).

11.3 Colinear solutions

In the next chapter, the approach described above will be used to derive new solutions for colliding gravitational waves whose polarization vectors are not aligned. Such solutions essentially have W non-zero, and hence Z and E are complex. However, before moving on to consider such cases, it is appropriate first to review the colinear solutions described in previous chapters. These solutions have Z and E real.

(i) The solution of Khan and Penrose (1971), discussed in Chapter 3 and Section 8.2, which describes the collision of aligned impulsive gravitational waves, is given by

$$Z = \frac{(1 + t)}{(1 - t)}, \qquad \text{or} \qquad E = t. \tag{11.28}$$

In this case $f = \frac{1}{2} - u^2$ and $g = \frac{1}{2} - v^2$. It may be observed that, for stationary axisymmetric space-times, (11.28) is the Ernst potential which leads to the Schwarzschild solution.

(ii) The Szekeres (1972) class of solutions, described in Chapter 9, is given by

$$Z = \left(\frac{1+t}{1-t}\right)^{(k_1+k_2)/2} \left(\frac{1+z}{1-z}\right)^{(k_1-k_2)/2}. \tag{11.29}$$

In this case $f = \frac{1}{2} - (au)^{n_1} + \ldots$ and $g = \frac{1}{2} - (bv)^{n_2} + \ldots$, with $k_i^2 = 2(1 - 1/n_i)$ for $i = 1, 2$ and $n_i \geq 2$.

(iii) The 'solution' of Stoyanov (1979) given by (10.24) uses

$$Z = (1-t^2)^{a/2}(1-z^2)^{a/2}. \tag{11.30}$$

In Section 10.2 it has been argued that this solution must be considered to be unphysical on its own, but it may be included as a factor in more general solutions. It does not satisfy the boundary conditions (11.26).

(iv) The solution of Ferrari and Ibañez (1987*a*) and Griffiths (1987), which is described in Section 10.3, is characterized by

$$\begin{aligned} Z &= \left(\frac{1+t}{1-t}\right)^a e^{b\sigma}, \\ \text{where} \qquad \sigma &= (3z^2-1)\left(\tfrac{1}{4}(3t^2-1)\log\left(\frac{1+t}{1-t}\right) - \tfrac{3}{2}t\right). \end{aligned} \tag{11.31}$$

In this case $f = \frac{1}{2} - (c_1 u)^n + \ldots$ and $g = \frac{1}{2} - (c_2 v)^n + \ldots$, where $n \geq 2$ and $(a+b)^2 = 2(1-1/n)$. It may be observed that, when $a = 1$, (11.31) is the Ernst potential which, for stationary axisymmetric solutions, leads to the solution of Erez and Rosen (1959) which describes the external field of a non-rotating body with a quadrupole moment.

(v) The generalized solution of Ferrari and Ibañez (1987*b*), described in Section 10.4, is characterized by

$$Z = \left(\frac{1+t}{1-t}\right)^k (1-t^2)^{a/2}(1-z^2)^{a/2}. \tag{11.32}$$

This can be seen to include the Stoyanov factor (11.30), and to reduce to a Szekeres solution with $k_1 = k_2 = k$ when $a = 0$. Again $f = \frac{1}{2} - (c_1 u)^n + \ldots$ and $g = \frac{1}{2} - (c_2 v)^n + \ldots$, where $n \geq 2$ and $k^2 = 2(1-1/n)$. The degenerate cases occur when $k = 1$ and $a = \pm 1$.

(vi) The solution of Tsoubelis and Wang (1989) given by (10.44) has

$$Z = \left(\frac{1+t}{1-t}\right)^{b_1/2} \left(\frac{1+z}{1-z}\right)^{b_2/2} (1-t^2)^{a/2}(1-z^2)^{a/2}. \tag{11.33}$$

This can be seen to be a generalization of (11.29) and (11.32), and its properties can immediately be deduced.

(vii) The odd order solution of Griffiths (1987) described in Section 10.6 uses the Ernst potential

$$Z = e^{-azQ_1(t)}, \qquad \text{where} \qquad Q_1(t) = \frac{t}{2}\log\left(\frac{1+t}{1-t}\right) - 1. \tag{11.34}$$

Again $f = \frac{1}{2} - (c_1u)^n + \ldots$ and $g = \frac{1}{2} - (c_2v)^n + \ldots$, where $n \geq 2$ and $a^2 = 8(1-1/n)$.

It is not difficult to see how further solutions of this type can be generated.

12

SOLUTION-GENERATING TECHNIQUES

For stationary axisymmetric space-times many solution-generating techniques are known. These are associated with the two Killing vectors, and with the internal symmetries of the Ernst equation. Colliding plane wave solutions also have two Killing vectors, and the main field equations can be transformed exactly to the Ernst equation. The main difference is that the solutions of the Ernst equation now contain the metric functions explicitly rather than the field potentials as in the stationary axisymmetric case. It is therefore to be expected that many of the familiar solution generating techniques can be used to obtain new colliding plane wave solutions from already known, or 'seed', solutions.

In this chapter we will consider only vacuum solutions. This will enable us to continue to concentrate on the collision of gravitational waves. Various techniques will be discussed here, and the main solutions that have been obtained using them will be described in the next chapter.

Many techniques are also known by which solutions of the Einstein–Maxwell equations, and other non-vacuum solutions, can be generated from known vacuum solutions. These will be discussed later in Chapter 15.

12.1 The colinear case

Consider first the case when the approaching waves have colinear polarization. The metric can be taken in the form of the line element (6.20) with $W = 0$. In this case, given any solution U_o, V_o, M_o of the field equations (6.22), then another solution of the same equations again with $W = 0$ is given by U, V and M, where

$$\begin{aligned} U &= U_o = -\log(f(u) + g(v)) \\ V &= V_o + aU_o \\ M &= M_o + aV_o + \tfrac{1}{2}a^2U_o \end{aligned} \tag{12.1}$$

where a is an arbitrary constant.

This transformation is well known (Halilsoy, 1985). It can clearly be seen that, if the initial solution satisfies the boundary conditions appropriate for colliding waves, then so does the new solution. It can also

be seen that the transformation (12.1) is equivalent to adding to V the Stoyanov solution (10.29).

It may also be observed that the transformation (12.1) can be used to obtain the soliton solution of Ferrari and Ibañez (1987*b*) described in Section 10.4, using as seed the Khan–Penrose solution. Ferrari and Ibañez, however, originally obtained their solution using the soliton techniques mentioned in Section 12.6.

Another transformation which maps one colinear solution of (6.22) on another, whether or not they satisfy the boundary conditions, is given by

$$\begin{aligned} V &= bV_{\rm o} \\ M &= (b^2-1)\log f'g' + \tfrac{1}{2}(b^2-1)U + b^2 M_{\rm o} + \text{const} \end{aligned} \tag{12.2}$$

where b is an arbitrary constant and $e^{-U} = f(u) + g(v)$.

When $b \neq \pm 1$, it can be seen that the transformation (12.2) is singular on the boundaries of the interaction region where $f' = 0$ and $g' = 0$. It follows that the necessary boundary conditions cannot be satisfied for both solutions, if the same expressions for f and g are used. If the seed solution satisfies the boundary conditions as described in Chapter 7, with $U_{\rm o} = -\log(f_{\rm o} + g_{\rm o})$ and $M_{\rm o}$ containing the terms

$$M_{\rm o} = \ldots + k_{\rm o1}\log(\tfrac{1}{2} - f_{\rm o}) + k_{\rm o2}\log(\tfrac{1}{2} - g_{\rm o}) + \ldots \tag{12.3}$$

with

$$f_{\rm o} = \tfrac{1}{2} - (c_{\rm o1}u)^{n_{\rm o1}} + \ldots, \qquad g_{\rm o} = \tfrac{1}{2} - (c_{\rm o2}v)^{n_{\rm o2}} + \ldots \tag{12.4}$$

then the new solution containing the terms (7.10) will also satisfy the boundary conditions if f and g are replaced by

$$f = \tfrac{1}{2} - (c_1 u)^{n_1} + \ldots, \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} + \ldots \tag{12.5}$$

where

$$1 - \frac{1}{n_1} = b^2\left(1 - \frac{1}{n_{\rm o1}}\right), \qquad 1 - \frac{1}{n_2} = b^2\left(1 - \frac{1}{n_{\rm o2}}\right) \tag{12.6}$$

provided $n_1 \geq 2$ and $n_2 \geq 2$. These equations give the new expression for U in terms of the null coordinates u and v. It may be noted that the inequalities (12.6) provide strong constraints on the range of permissible values of the parameter b in (12.2).

It can thus be seen that this transformation with $b^2 \neq 1$ may be used to change the profile of the approaching waves. For example, it could

be used to derive the Szekeres solution described in Chapter 9 from the Khan–Penrose solution. The two transformations (12.1) and (12.2) may also be used in conjunction.

It may also be noted that the transformations for V in (12.1) and (12.2) may be restated in the form that, if a real Z_o is a solution of Ernst's equation (11.18), then

$$Z = (1-t^2)^{a/2}(1-z^2)^{a/2} Z_\mathrm{o} \tag{12.7}$$

and

$$Z = Z_\mathrm{o}^{\,b} \tag{12.8}$$

are also real solutions with arbitrary constants a and b, though as explained above, the possible values of b are constrained by the boundary conditions.

These results are, of course, implicitly contained in the general approach to colinear solutions described in Section 10.1. The point to note is that, when the approaching waves have constant aligned polarization so that Z and E are real and $W=0$, the main field equations can be linearized. In this case a general superposition of solutions is possible, as illustrated in (10.16) together with (10.20). Further generating techniques for this case have been described by Kitchingham (1984).

In the more general case, however, Z and E are complex, and the field equations are essentially non-linear. In this situation, the generating techniques described in the remainder of this chapter are particularly significant.

12.2 Rotations and Ehlers transformations

Consider first the well known result that, if E_o is a solution of Ernst's equation (11.14), then

$$E = e^{i\alpha} E_\mathrm{o} \tag{12.9}$$

where α is a constant, is also a solution.

In terms of the alternative function Z the equivalent result is that, if Z_o is a solution of (11.8), then another solution is given by

$$Z = \frac{\cos\frac{\alpha}{2}\, Z_\mathrm{o} - i\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\, Z_\mathrm{o}}. \tag{12.10}$$

By applying the coordinate rotation

$$x + iy = e^{i\alpha/2}(x_\mathrm{o} + iy_\mathrm{o}) \tag{12.11}$$

to the line element (11.12), it may be shown that this rotation is exactly equivalent to the transformation (12.9), indicating that this transformation may be simply interpreted as a global rotation of coordinates.

In terms of the metric functions of the line element (6.20), this rotation can be shown to be equivalent to the result that, if U_o, V_o, W_o and M_o are solutions of (6.22), then another solution is given by

$$\begin{aligned} U &= U_o \\ \tanh V &= \cos\alpha \tanh V_o + \sin\alpha \,\mathrm{sech} V_o \tanh W_o \\ \sinh W &= \cos\alpha \sinh W_o - \sin\alpha \sinh V_o \cosh W_o \\ M &= M_o. \end{aligned} \tag{12.12}$$

It may be noted that, in (12.12), the function M is unaltered. Thus the boundary conditions are not affected, and so neither are the functions $f(u)$ and $g(v)$. In fact, it is clear that (12.12) does not generate a new solution, but merely a rotation of the original.

It can also be shown that, if the seed solution is colinear so that we can put $W_o = 0$, then the transformation (12.12) reduces to the transformation suggested by Ray (1980) as corrected by Halilsoy (1981).

Another well known result is that, if Z_o is a solution of Ernst's equation (11.8), then

$$Z = \frac{Z_o}{1 - icZ_o} \tag{12.13}$$

where c is a real constant, is also a solution. This is loosely referred to as an Ehlers transformation (Ehlers, 1957).

Equivalently, if E_o is a solution of (11.14), then another solution is given by

$$1 + E = \frac{1 + E_o}{1 - \frac{ic}{2}(1 + E_o)}. \tag{12.14}$$

In terms of the metric functions in (6.20), we may start with the solutions U_o, V_o, W_o and M_o of (6.22). After integrating (11.21), it can then be shown that another solution is given by

$$\begin{aligned} U &= U_o \\ e^{2V} &= e^{2V_o} + 2ce^{V_o} \tanh W_o + c^2 \\ \sinh W &= \sinh W_o + c \cosh W_o e^{-V_o} \\ M &= M_o. \end{aligned} \tag{12.15}$$

Since the metric function M is unchanged, it follows that the boundary conditions for the new solution are satisfied if, and only if, they are satisfied for the seed solution.

The above transformation is also related to the well known result that, if Z_o is a solution of Ernst's equation (11.8), then another solution is given by

$$Z = Z_\mathrm{o} + ib \tag{12.16}$$

where b is a real constant. Equivalently, if E_o is a solution of (11.14), then another solution is given by

$$1 - E = \frac{1 - E_\mathrm{o}}{1 + \frac{ib}{2}(1 - E_\mathrm{o})}. \tag{12.17}$$

In terms of the metric functions in (6.20), this leads to the transformation that, if U_o, V_o, W_o and M_o are solutions of (6.22), then another solution is given by

$$\begin{aligned} U &= U_\mathrm{o} \\ e^{-2V} &= e^{-2V_\mathrm{o}} + 2be^{-V_\mathrm{o}} \tanh W_\mathrm{o} + b^2 \\ \sinh W &= \sinh W_\mathrm{o} + b \cosh W_\mathrm{o} e^{V_\mathrm{o}} \\ M &= M_\mathrm{o}. \end{aligned} \tag{12.18}$$

This can clearly be seen to be identical to (12.15) but with the sign of V interchanged.

All of the above transformations can be shown to be obtainable from a general rotation and rescaling of the coordinates in the x,y plane. This can be described by subjecting the Killing vectors ∂_x and ∂_y to any SL(2,R) transformation

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \qquad \text{where} \qquad ad - bc = 1. \tag{12.19}$$

This leaves U and M invariant but, by considering the line element (11.6), the Ernst function Z can be shown to transform as

$$Z = i\frac{(aZ_\mathrm{o} + ib)}{(cZ_\mathrm{o} + id)}. \tag{12.20}$$

The transformation (12.20) clearly incorporates (12.10), (12.13) and (12.16) which include both pure rotations and those referred to as Ehlers transformations. When applied to metric functions on its own this transformation clearly has little physical significance. It follows that these transformations on their own can not be used to generate genuinely non-colinear solutions from colinear ones. However, they may be combined with some other method to obtain such solutions.

In particular these transformations may be applied to the potentials for the metric functions that will be described in Section 12.4. Strictly speaking it is only these transformations, applied in the potential space, that should be referred to as Ehlers transformations (Ehlers, 1957). It is then possible that a sequence both of coordinate transformations (12.19) and Ehlers transformations (12.20) in the potential space can be applied alternately to generate a further class of solutions with an infinite number of parameters.

12.3 Geroch transformations

Geroch (1971) has presented a general technique for generating solutions of Einstein's source-free equations from known solutions which possess a Killing vector ξ^μ. As originally presented, the method is to consider a vacuum space-time with metric $g_{\mu\nu}$, and to solve the equations

$$\begin{aligned} \nabla_\kappa G &= \epsilon_{\kappa\lambda\mu\nu}\xi^\lambda \nabla^\mu \xi^\nu \\ \nabla_{[\kappa} P_{\lambda]} &= \tfrac{1}{2}\epsilon_{\kappa\lambda\mu\nu}\nabla^\mu\xi^\nu \\ \nabla_{[\kappa} Q_{\lambda]} &= 2F\nabla_\kappa \xi_\lambda + G\epsilon_{\kappa\lambda\mu\nu}\nabla^\mu\xi^\nu \end{aligned} \tag{12.21}$$

for G, P_μ and Q_μ subject to the following conditions:

$$\xi^\mu \xi_\mu = F\,, \qquad \xi^\mu P_\mu = G\,, \qquad \xi^\mu Q_\mu = F^2 + G^2 - 1. \tag{12.22}$$

(A solution of these equations is known to exist.) Then, the basic result is that a new solution of the source-free equations, depending on an arbitrary real parameter α, is given by

$$g'_{\mu\nu} = F\tilde{F}^{-1}\left(g_{\mu\nu} - F^{-1}\xi_\mu\xi_\nu\right) + \tilde{F}\eta_\mu\eta_\nu \tag{12.23}$$

where

$$\begin{aligned} \tilde{F} &= F\left((\cos\alpha - G\sin\alpha)^2 + F^2\sin^2\alpha\right)^{-1} \\ \eta_\mu &= \tilde{F}^{-1}\xi_\mu + 2\sin\alpha\cos\alpha P_\mu - \sin^2\alpha Q_\mu. \end{aligned} \tag{12.24}$$

It can be seen that F and G are respectively the norm and twist potential of the Killing field ξ^μ.

This technique determines a new solution with one extra parameter α. A second application of the method simply yields another member of the same one-parameter family of solutions. It may be observed that the new space-times also have the same Killing vector as the seed solution.

It may also be shown that these transformations include both the Ehlers transformation and the rotation described in previous sections.[1]

The case when there exist two commuting Killing vectors, which is the case for colliding plane waves, has also been considered (Geroch, 1972). Clearly, the above technique may be applied using any linear combination of the Killing vectors. Successive applications of the method using different combinations of the Killing vectors yield metrics which depend on the order of the vectors used. The resulting class of solutions depends on two arbitrary functions, and retains the same pair of commuting Killing vectors. The algebraic structure of these transformations has been analysed by Geroch (1972).

The Geroch transformations, which relate one vacuum solution to another, in fact form an infinite-dimensional group. The first really useful realization of the Lie algebra of the Geroch group was formulated by Kinnersley and Chitre (1977, 1978*a,b*), who demonstrated the action of the infinitesimal elements of the group in terms of an infinite hierarchy of potentials. Kinnersley and Chitre, and also Hoenselaers, Kinnersley and Xanthopoulos (1979), exploited this approach to derive new stationary axisymmetric solutions. This formalism has been extended to space-times with two space-like Killing vectors by Kitchingham (1984), although he initially applied it only in the cosmological context.

Hauser and Ernst (1979, 1980) introduced a realization of the finite elements of the Geroch group. They also showed that the Kinnersley–Chitre transformations can be carried out by solving an appropriate homogeneous Hilbert problem.

A simpler realization $G_0(\Sigma_1)$ of the Geroch group of transformations which is specifically applied to colliding wave solutions has been found by Ernst, García-Díaz and Hauser (1987*b*). With this, colliding wave solutions with any desired number of parameters may be constructed.

When applying the original transformation (12.23) to colliding plane waves, it is convenient to start first with the Szekeres line element (6.20), in which the two Killing vectors are $\xi^{(1)\mu} = \delta^\mu_2$ and $\xi^{(2)\mu} = \delta^\mu_3$. It is then appropriate to consider a general Killing vector

$$\xi^\mu = a\delta^\mu_2 + b\delta^\mu_3 \tag{12.25}$$

where a and b are arbitrary constants. It follows immediately that

$$F = e^{-U}(a^2e^V \cosh W - 2ab\sinh W + b^2e^{-V}\cosh W). \tag{12.26}$$

[1] For a general introduction to Geroch and related transformations see Xanthopoulos (1985).

It is then possible to put two of the components of P_μ and Q_μ each equal to zero. Putting

$$P_\mu = (0, 0, P_2, P_3)\,, \qquad Q_\mu = (0, 0, Q_2, Q_3), \tag{12.27}$$

the remaining components of P_μ must then satisfy the equations

$$\begin{aligned}
P_{2,u} &= -e^{-U}\left(ae^{V}(W_u - V_u \sinh W \cosh W) - b(U_u + V_u \cosh^2 W)\right)\\
P_{2,v} &= e^{-U}\left(ae^{V}(W_v - V_v \sinh W \cosh W) - b(U_v + V_v \cosh^2 W)\right)\\
P_{3,u} &= -e^{-U}\left(a(U_u - V_u \cosh^2 W) - be^{-V}(W_u + V_u \sinh W \cosh W)\right)\\
P_{3,v} &= e^{-U}\left(a(U_v - V_v \cosh^2 W) - be^{-V}(W_v + V_v \sinh W \cosh W)\right).
\end{aligned} \tag{12.28}$$

These equations are automatically integrable in view of (6.22). It is then possible to put

$$G = aP_2 + bP_3 \tag{12.29}$$

and to choose the components of Q_μ such that

$$aQ_2 + bQ_3 = F^2 + G^2 - 1. \tag{12.30}$$

Equations (12.21) are then satisfied.

The new metric is then given by (12.23). The expressions for the new functions are rather involved. However, it is clear that

$$e^{-M} = \left((\cos\alpha - G\sin\alpha)^2 + F^2\sin^2\alpha\right)e^{-M_\circ} \tag{12.31}$$

which indicates that the boundary conditions are satisfied for the new solution, if they were satisfied for the seed solution.

In the case when the initial solution has colinear polarization, this method can be considerably simplified. Such a situation was considered by Panov (1979*b*), and will be discussed further in Section 13.2.

12.4 The Neugebauer–Kramer involution

Chandrasekhar and Ferrari (1984) have shown that the main field equations for colliding plane waves can be written in the form of Ernst's equation (11.18) involving the complex function Z. In this case, the real and imaginary parts of this function, denoted by $Z = \chi + i\omega$, are the metric coefficients given by (11.2–4). Using these components, the main field equations can be written as

$$\left(\frac{(1-t^2)}{\chi}\chi_t\right)_{,t} - \left(\frac{(1-z^2)}{\chi}\chi_z\right)_{,z} + \frac{(1-t^2)}{\chi^2}\omega_t^2 - \frac{(1-z^2)}{\chi^2}\omega_z^2 = 0 \tag{12.32}$$

and

$$\left(\frac{(1-t^2)}{\chi^2}\omega_t\right)_{,t} - \left(\frac{(1-z^2)}{\chi^2}\omega_z\right)_{,z} = 0. \tag{12.33}$$

These are respectively the real and imaginary parts of (11.18).

The second of these equations immediately implies that there exists a function Φ such that

$$\Phi_z = \frac{(1-t^2)}{\chi^2}\omega_t, \qquad \Phi_t = \frac{(1-z^2)}{\chi^2}\omega_z. \tag{12.34}$$

It follows that Φ must satisfy the equation

$$\left(\frac{\chi^2}{(1-z^2)}\Phi_t\right)_{,t} - \left(\frac{\chi^2}{(1-t^2)}\Phi_z\right)_{,z} = 0. \tag{12.35}$$

At this point, it is convenient to introduce a new function Ψ defined by

$$\Psi = \sqrt{1-t^2}\sqrt{1-z^2}\,\chi^{-1}. \tag{12.36}$$

From (11.1) and (10.11), it may be seen that $-\Psi$ is the coefficient of dx^2 in the line element (11.4).

With this definition and (12.34), the equations (12.32) and (12.35) can now be rewritten in the form

$$\left(\frac{(1-t^2)}{\Psi}\Psi_t\right)_{,t} - \left(\frac{(1-z^2)}{\Psi}\Psi_z\right)_{,z} + \frac{(1-t^2)}{\Psi^2}\Phi_t^2 - \frac{(1-z^2)}{\Psi^2}\Phi_z^2 = 0 \tag{12.37}$$

and

$$\left(\frac{(1-t^2)}{\Psi^2}\Phi_t\right)_{,t} - \left(\frac{(1-z^2)}{\Psi^2}\Phi_z\right)_{,z} = 0. \tag{12.38}$$

These equations may be seen to be identical to (12.32) and (12.33) with Ψ and Φ replacing χ and ω. Thus, if $Z_o = \chi + i\omega$ is a solution of (11.18), then another solution is given by

$$Z = \Psi + i\Phi \tag{12.39}$$

where Ψ is given by (12.36), and Φ is chosen to satisfy (12.34). This correspondence is known as the Neugebauer–Kramer involution. It is known to map real metrics to real metrics only for space-times with two space-like Killing vectors. In modern terminology, it is clear that the equations (12.34) provide an auto-Bäcklund transformation for the Ernst equation (11.18).

In order to obtain the remaining metric function M it is necessary to integrate equations (7.9), which may be rewritten in the form

$$\begin{aligned} S_f &= -\frac{1}{2}\,\frac{(f+g)}{\chi^2}\,(\chi_f^2+\omega_f^2) \\ S_g &= -\frac{1}{2}\,\frac{(f+g)}{\chi^2}\,(\chi_g^2+\omega_g^2) \end{aligned} \tag{12.40}$$

which is equivalent to (11.21). In terms of the alternative functions Ψ and Φ these become

$$\begin{aligned} S_f &= -\frac{1}{2(f+g)}+\frac{\Psi_f}{\Psi}-\frac{1}{2}\,\frac{(f+g)}{\Psi^2}\,(\Psi_f^2+\Phi_f^2) \\ S_g &= -\frac{1}{2(f+g)}+\frac{\Psi_g}{\Psi}-\frac{1}{2}\,\frac{(f+g)}{\Psi^2}\,(\Psi_g^2+\Phi_g^2). \end{aligned} \tag{12.41}$$

It follows from this that, for any solution with $Z_{\rm o}=\chi+i\omega$ and $M_{\rm o}$, there exists another solution with $Z=\Psi+i\Phi=Z_{\rm o}$ and M where

$$M=M_{\rm o}+\tfrac{1}{2}U_{\rm o}+\log\chi. \tag{12.42}$$

This immediately indicates that, if the initial solution satisfies the boundary conditions for colliding plane waves, then so also does the new solution, provided

$$U=U_{\rm o}. \tag{12.43}$$

A second application of this technique only leads back to the original solution, apart from an arbitrary complex constant. The presence of such a constant has already been considered in Section 12.2 following equation (12.16).

It may be noted in passing that, in the colinear case in which $\omega=0$ and $\Phi=0$, this generation technique is equivalent to putting $V=U_{\rm o}-V_{\rm o}$ which is effectively contained in (12.1) but with a change in the sign of V. This, for example, would generate the Ferrari–Ibañez solution from the Khan–Penrose solution.

It is of interest to observe that, for stationary axisymmetric space-times, the main field equations can be written in the form of Ernst's equation (11.8), whose solution $Z=\Psi+i\Phi$ is considered as a complex potential for the metric functions χ and ω that are obtained from (12.36) and (12.34). In contrast, for colliding plane waves, both the potential (12.39) and the metric function (11.5) satisfy the same equation. Thus, in this case, the Neugebauer–Kramer involution is equivalent to a generation

technique in which any solution has a dual solution in which the metric functions and the associated potentials are interchanged.

12.5 A combined transformation

In a recent paper, Halilsoy (1988*b*) has considered a technique which he applied initially in the context of colliding shock electromagnetic waves (Halilsoy, 1988*a*). This can be described as follows.

If V_o is a solution of the vacuum equations (9.3), (10.2) or (10.13) corresponding to a collision of colinear gravitational waves (with $W_o = 0$), then a noncolinear solution is given in terms of the function E satisfying (11.14) by

$$E = Pe^{i\theta} \tag{12.44}$$

where

$$\begin{aligned} P^2 &= \frac{\cosh aV_o - \cos\alpha}{\cosh aV_o + \cos\alpha} \\ \tan\theta &= -\tan\alpha \coth aV_o \end{aligned} \tag{12.45}$$

where α is an arbitrary constant. The original solution is obtained when $\alpha = 0$ and $a = 1$.

In terms of the alternative function Z, the original solution may be described by a real function Z_o and the new solution satisfying (11.8) is then given by

$$Z = \frac{i\tan\frac{\alpha}{2} + Z_o^a}{1 - i\tan\frac{\alpha}{2} Z_o^a}. \tag{12.46}$$

This can thus be seen to be a transformation of the type (12.2) followed by an Ehlers transformation in a slightly more general form than (12.13).

In terms of the metric functions of the Szekeres line element (6.20), the original solution is described by the three functions U_o, V_o and M_o which are each functions of f_o and g_o where, as in (7.8), it is convenient to put

$$e^{-U_o} = f_o + g_o, \qquad e^{-M_o} = \frac{f_o' g_o'}{\sqrt{f_o + g_o}} e^{-S_o} \tag{12.47}$$

and, as in (12.4),

$$f_o = \tfrac{1}{2} - (c_{o1}u)^{n_{o1}} + \ldots, \qquad g_o = \tfrac{1}{2} - (c_{o2}v)^{n_{o2}} + \ldots. \tag{12.48}$$

The new solution (12.44) is given by

$$\begin{aligned} \tanh V &= \cos\alpha \tanh aV_o \\ \sinh W &= \tan\alpha \cosh aV_o. \end{aligned} \tag{12.49}$$

With this, equations (7.7) can be integrated to give

$$M = (a^2 - 1)\log f_o' g_o' + \tfrac{1}{2}(a^2 - 1)U_o + a^2 M_o + \text{const} \tag{12.50}$$

which is similar to (12.2). Now, using the same arguments as in Section 12.1, the new solution will only satisfy the appropriate boundary conditions if the forms of the functions f_o and g_o are modified. Accordingly we now put

$$e^{-U} = f + g, \qquad e^{-M} = \frac{f'g'}{\sqrt{f+g}} e^{-S}. \tag{12.51}$$

The parameters f and g are now expressed in terms of the null coordinates u and v in the form

$$f = \tfrac{1}{2} - (c_1 u)^{n_1} + \ldots, \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} + \ldots \tag{12.52}$$

where

$$1 - \frac{1}{n_1} = a^2\left(1 - \frac{1}{n_{o1}}\right), \qquad 1 - \frac{1}{n_2} = a^2\left(1 - \frac{1}{n_{o2}}\right). \tag{12.53}$$

With this restriction, the boundary conditions are satisfied, and

$$S = a^2 S_o. \tag{12.54}$$

If, in the original solution, the functions f_o, g_o and M_o have been chosen such that the terms $f_o' g_o'$ in (12.47) have already cancelled the terms $k_{o1}\log(\frac{1}{2} - f_o)$ and $k_{o2}\log(\frac{1}{2} - g_o)$ in S_o, then the new solution for M will be given by

$$M = \tfrac{1}{2}(a^2 - 1)U_o + a^2 M_o. \tag{12.55}$$

The method described in this section has been used by Halilsoy (1988*b*) to obtain a class of solutions which appears to generalize the Szekeres solutions. However, since it has been obtained by a real transformation followed by an Ehlers transformation, as described at the end of Section 12.2 it is essentially simply a rotation and rescaling of the Szekeres solutions. It does not therefore describe the collision of genuinely non-aligned gravitational waves.

12.6 Other methods

Within the context of stationary axisymmetric space-times, many techniques have been developed in recent years by which new solutions may

be obtained from initial vacuum solutions.[2] Most of these methods can be applied also to the colliding plane wave situation. The main difference being that, in this case, the solutions of Ernst's equation may themselves either be metric functions or potentials for those functions as described in Section 12.4.

The other difference relates to the change of boundary conditions. For stationary axisymmetric solutions, it is usual to impose the condition that the space-times are asymptotically flat, and techniques for generating new solutions of Ernst's equation in which the space-time satisfies this condition have been developed. One of the most general of these is that of Cosgrove (1977), which generalizes the Tomimatsu–Sato solutions to arbitrary continuous deformation parameter δ. However, for colliding plane waves, the boundary conditions are of a totally different character, and many of the solutions of Ernst's equation that are appropriate for axisymmetric fields, such as the Tomimatsu–Sato solutions, are now found to be inconsistent with the boundary conditions for colliding plane waves.

Hassan, Feinstein and Manko (1990) have adapted a generating technique of Gutsunaev and Manko (1988) for stationary axisymmetric space-times to derive solutions which describe the interaction following the collision of plane gravitational waves. This technique generates solutions in which the approaching waves have variable polarization from a 'seed' solution V_o of the equation (10.13) for the colinear case. A solution of the Ernst equation (11.17) is then given by

$$Z = e^{V_o} \frac{(1-\alpha^2 ab)t + i\alpha(a+b)z - (1+i\alpha a)(1-i\alpha b)}{(1-\alpha^2 ab)t + i\alpha(a+b)z + (1+i\alpha a)(1-i\alpha b)} \tag{12.56}$$

where α is a constant and the functions $a(t,z)$ and $b(t,z)$ are obtained by integrating the equations

$$\begin{aligned}
(\log a)_{,t} &= \frac{1}{t-z}[(tz-1)V_{o,t} + (1-z^2)V_{o,z}] \\
(\log a)_{,z} &= \frac{1}{t-z}[(1-t^2)V_{o,t} + (tz-1)V_{o,z}] \\
(\log b)_{,t} &= \frac{1}{t+z}[(tz+1)V_{o,t} + (1-z^2)V_{o,z}] \\
(\log b)_{,z} &= \frac{1}{t+z}[(1-t^2)V_{o,t} + (tz+1)V_{o,z}].
\end{aligned} \tag{12.57}$$

The boundary conditions then have to be applied.

[2] For recent reviews see Kramer (1987), Dodd (1990) and the conference proceedings edited by Hoenselaers and Dietz (1984).

One particularly important technique that has been developed in recent years is the inverse scattering method of Belinskii and Zakharov (1978, 1979) which had previously been developed as a soliton technique. This method has been applied by Carr and Verdaguer (1983) to plane-symmetric space-times, and their results have been interpreted in terms of cosmological gravitational waves. The relation between this and other techniques has been analysed by Letelier (1987).

The first solution for colliding plane gravitational waves that was obtained using the inverse scattering method was that of Ferrari and Ibañez (1987*b*), who obtained the solution described in Section 10.4 using the method in the form developed by Carr and Verdaguer (1983). In this paper they took as seed metric the Kasner solution (10.26) which is equivalent to the Stoyanov solution (10.29). It may therefore be noted that, although the new solution satisfies the boundary conditions for colliding plane waves, the seed solution in this case does not. Non-aligned solutions have also been obtained using this method by Ferrari, Ibañez and Bruni (1987*b*).

Another solution-generating technique that has been developed in recent years is that associated with Bäcklund transformations. These were first applied to general relativity independently by Harrison (1978) and Neugebauer (1979). Bäcklund transformations can also be described as double-Harrison transformations, or as quadruple-Neugebauer transformations. The explicit relation between this technique and other methods has been discussed by Cosgrove (1980, 1982).

The effect of a double-Harrison transformation can be deduced by any number of competing techniques. Hauser and Ernst (1979, 1980) developed a method which involves solving a homogeneous Hilbert problem. Cosgrove (1981) has shown how Bäcklund transformations can be conveniently expressed in this formalism. The method has been applied to colliding wave problems by Ernst, García-Díaz and Hauser (1988) (see also Ernst, 1988), and a class of new colliding wave solutions has been obtained.

It has been observed by Witten (1979) that Einstein's vacuum equations for stationary, axisymmetric space-times are equivalent to a form of the self-dual Yang–Mills equations. He therefore suggested that it should be possible to generate solutions of Einstein's equations by using twistor methods to construct self-dual Yang–Mills fields with appropriate symmetries. Such an approach to stationary axisymmetric space-times has subsequently been developed by Ward (1983), Woodhouse (1987) and also by Woodhouse and Mason (1988), who showed how this approach relates to the other solution-generating techniques mentioned above. It follows that this approach may also be applied for other types of space-

times which contain two commuting Killing vectors. Woodhouse (1989) has applied these techniques to generate cylindrical gravitational waves. Twistor techniques may also be developed to generate colliding plane wave solutions.

Any of the above techniques may be used to generate new solutions of Ernst's equation. Most have been applied mainly to stationary axisymmetric space-times. They may, however, also be used to generate solutions of Ernst's equation that can be considered as possible to colliding plane wave solutions. The difficulty is then to find under what circumstances the appropriate boundary conditions are satisfied. Whether or not a new solution in region IV satisfies these conditions will generally depend both on the particular technique that is employed and also on the initial seed solution.

13

VACUUM SOLUTIONS WITH NON-ALIGNED POLARIZATION

The purpose of this chapter is to describe the known exact solutions for colliding gravitational waves in which the polarization of the approaching waves is not aligned. The first solution of this type was obtained by Nutku and Halil (1977). A further generalization of this solution was attempted by Halil (1979), but this has subsequently been found to be incorrect and so will not be considered here.

13.1 The Nutku–Halil solution

The Khan–Penrose solution discussed in Chapter 3 describes the collision of impulsive waves whose polarization vectors are aligned. Nutku and Halil (1977) have generalized this solution to give one which describes the collision of impulsive gravitational waves with non-colinear polarization. This is the most simple solution of this type.

In terms of the Szekeres line element (6.20), the metric functions of this solution can be written in the form

$$
\begin{aligned}
e^{-U} &= 1-u^2-v^2 \\
e^{-V} &= \sqrt{\frac{(1-E)(1-\bar{E})}{(1+E)(1+\bar{E})}} \\
\sinh W &= -\frac{i(E-\bar{E})}{1-E\bar{E}} \\
e^{-M} &= \frac{1-E\bar{E}}{\sqrt{1-u^2}\sqrt{1-v^2}\sqrt{1-u^2-v^2}}
\end{aligned}
\tag{13.1}
$$

where

$$
E = e^{i\alpha}u\sqrt{1-v^2} + e^{i\beta}v\sqrt{1-u^2} \tag{13.2}
$$

and α and β are constants such that $(\alpha-\beta)$ is the angle between the polarization vectors of the approaching waves. This solution can be seen to reduce to the Khan–Penrose solution (3.9) when $\alpha=\beta=0$. It may also be observed that the functions f and g are given by $f=\frac{1}{2}-u^2$, and $g=\frac{1}{2}-v^2$.

This solution has been analysed in great detail by Chandrasekhar and Ferrari (1984). Their approach commences by rewriting the field equations in the form of Ernst's equation as described in Chapter 11. It can then be seen from (11.20) that E is the associated Ernst function. Chandrasekhar and Ferrari were then able to show that the Nutku–Halil solution is obtained by the simple choice of Ernst function given by

$$E = pt + iqz \tag{13.3}$$

where p and q are real constants which satisfy the condition $p^2 + q^2 = 1$. It is always possible to use a rotation of the type (12.9) to put $\beta = -\alpha$, and in this case $p = \cos\alpha$ and $q = \sin\alpha$.

It can easily be shown that the Ernst function Z associated with this solution is given by

$$Z = \frac{1 + t\cos\alpha + iz\sin\alpha}{1 - t\cos\alpha - iz\sin\alpha}. \tag{13.4}$$

It may be observed that (13.3) is in fact the Ernst potential which leads to the Kerr solution for stationary axisymmetric space-times. This reduces to the potential for the Schwarzschild solution when $q = 0$ or $\alpha = 0$. Thus the difference in the polarization of the approaching waves in this situation can be seen to be equivalent to the rotational parameter in the Kerr solution.

Taking the Ernst functions as (13.3) or (13.4), it can be shown that the boundary conditions described in Section 7.2 can only be satisfied if $f = \frac{1}{2} - (c_1 u)^2 + \ldots$, and $g = \frac{1}{2} - (c_2 v)^2 + \ldots$. It is then convenient to transform the null coordinates such that $f = \frac{1}{2} - u^2$, and $g = \frac{1}{2} - v^2$. Thus the form (13.2) is obtained uniquely from the Ernst function (13.3). It also follows from this that the approaching waves necessarily contain impulsive components.

Expressions for the components of the Weyl tensor Ψ_0, Ψ_2 and Ψ_4 describing the wave and interaction components have been given by Chandrasekhar and Ferrari (1984), though in a notation which differs slightly from that adopted here. However, it can still be clearly seen that this solution is a generalization of the Khan–Penrose solution in which the approaching impulsive waves have different polarization. Apart from the initial impulsive waves on the boundaries $u = 0$ and $v = 0$, the interiors of regions II and III are flat.

It can also be seen that the singularity structure of this solution is the same as that of the Khan–Penrose solution. There is a curvature singularity on the space-like surface $u^2 + v^2 = 1$ in region IV, and there

are fold singularities in regions II and III which are identical to those described in Section 8.3.

It may also be pointed out that Ernst (1986) has shown that that the Nutku–Halil solution can also be obtained using a double-Harrison (Bäcklund) transformation, with the seed metric being the isotropic Kasner metric.

13.2 The Panov solution

Panov (1979*b*) has applied the generation method of Geroch (1971), described in Section 12.3, to obtain a non-colinear solution from a colinear one.

In terms of the metric functions in (6.20), Panov started with a colinear solution $U_\circ$, $V_\circ$, and $M_\circ$ of (6.22) with $W_\circ = 0$. He then worked with the Killing vector $\xi^\mu = \delta^\mu_2$, which is (12.25) with $a = 1$ and $b = 0$. This enabled him to put $P_2 = 0$, and hence $\omega = 0$, and also from (12.30), $Q_2 = F^2 - 1$ and $Q_3 = 0$, where $F^2 = e^{V_\circ - U_\circ}$.

With these expressions, the metric components may be derived from (12.23), giving

$$\begin{aligned} U &= U_\circ \\ e^{-2V} &= e^{-2V_\circ}\Big(\cos^2\alpha + \sin^2\alpha\, e^{2(V_\circ - U_\circ)}\Big)^2 + \sin^2 2\alpha\, P_3^2 \\ \sinh W &= \sin 2\alpha\, P_3 e^{V_\circ}\Big(\cos^2\alpha + \sin^2\alpha\, e^{2(V_\circ - U_\circ)}\Big)^{-1} \\ e^{-M} &= e^{-M_\circ}\Big(\cos^2\alpha + \sin^2\alpha\, e^{2(V_\circ - U_\circ)}\Big) \end{aligned} \tag{13.5}$$

where, from (12.28), P_3 must satisfy

$$P_{3,u} = e^{-U_\circ}(V_{\circ,u} - U_{\circ,u})\ , \qquad P_{3,v} = -e^{-U_\circ}(V_{\circ,v} - U_{\circ,v}) \tag{13.6}$$

or, more conveniently,

$$P_{3,f} = (f+g)V_{\circ,f} + 1, \qquad P_{3,g} = -(f+g)V_{\circ,g} - 1. \tag{13.7}$$

It is worth pointing out that, if the initial Killing vector had been taken to be $\xi^\mu = \delta^\mu_3$, which is (12.25) with $a = 0$ and $b = 1$, an equivalent transformation would have been obtained with $P_3 = 0$ and $P_2 \neq 0$. This would have given an identical set of equations, but with V being replaced by $-V$, and P_3 by P_2.

In terms of the Ernst formulation of the problem described in Chapter 11, the above transformation implies that, if Z_o is a real solution of (11.18), then a new complex solution is given by

$$Z = \cos^2\alpha\, Z_o + (1-t^2)(1-z^2)\sin^2\alpha\, Z_o^{-1} + i\sin 2\alpha\, P_3 \qquad (13.8)$$

where $P_3(t,z)$ must satisfy the equations

$$\begin{aligned} P_{3,t} &= -(1-z^2)(\log Z_o)_{,z} - z \\ P_{3,z} &= -(1-t^2)(\log Z_o)_{,t} - t. \end{aligned} \qquad (13.9)$$

Panov then took the initial solution to be the totally general solution of Szekeres (1972), which can be considered to be equivalent to the general solution given here by (10.16). He has thus determined a general class of non-colinear solutions, and he was able to show that, when the appropriate boundary conditions have been applied, the approaching waves necessarily have variable polarization.

It has already been shown that the new solution obtained using a Geroch transformation will automatically satisfy the required boundary conditions if these are satisfied by the initial solution. In this case, it can also be seen that the new and initial solutions have the same singularity structure.

13.3 The Chandrasekhar–Xanthopoulos solution

Chandrasekhar and Ferrari (1984) have shown that the associated Ernst function which is contained in the Nutku–Halil solution can be written as $E_o = pt + iqz$ (13.3), where $p^2 + q^2 = 1$. Now, it is well known (see Chandrasekhar, 1983) that this is the Ernst potential that leads to the Kerr solution for stationary axisymmetric space-times. However, as described in Section 12.4, Chandrasekhar and Ferrari have also shown that, for the colliding plane wave situation, the Ernst function may be regarded either (a) as containing the metric functions, or (b) as a potential for those functions.

When the Ernst function (13.3) is considered as containing the metric functions, it leads to the Nutku–Halil solution described above. However, when the same function is considered as a potential, it then leads to the Kerr solution. Thus part of the the Kerr space-time must also be considered as a solution of the colliding plane wave equations, though in this situation the coordinates have a different interpretation.

This alternative interpretation of part of the Kerr space-time has been described in detail by Chandrasekhar and Xanthopoulos (1986*b*).

Essentially it may be considered to have been derived from the Nutku–Halil solution using the Neugebauer–Kramer involution described in Section 12.4.

We start here with the Ernst potential given by $E = pt + iqz$, so that

$$Z = \Psi + i\Phi = \frac{1 + pt + iqz}{1 - pt - iqz} \tag{13.10}$$

where $p^2 + q^2 = 1$. According to (12.36), the first metric function is then given by

$$\begin{aligned} \chi &= \sqrt{1-t^2}\sqrt{1-z^2}\,\Psi^{-1} \\ &= \sqrt{1-t^2}\sqrt{1-z^2}\left(\frac{(1-pt)^2 + q^2z^2}{1 - p^2t^2 - q^2z^2}\right) \end{aligned} \tag{13.11}$$

and, from (12.34), ω is given by

$$\omega_z = \frac{(1-t^2)}{\Psi^2}\,\Phi_t, \qquad \omega_t = \frac{(1-z^2)}{\Psi^2}\,\Phi_z \tag{13.12}$$

where

$$\Phi = \frac{2qz}{(1-pt)^2 + q^2z^2}. \tag{13.13}$$

With this, (13.12) becomes

$$\omega_z = \frac{4pq(1-t^2)(1-pt)z}{(1-p^2t^2-q^2z^2)^2}, \qquad \omega_t = \frac{2q(1-z^2)\big((1-pt)^2 - q^2z^2\big)}{(1-p^2t^2-q^2z^2)^2} \tag{13.14}$$

which can immediately be integrated to give

$$\omega = \frac{2}{q}\left(\frac{p(1-t^2)(1-pt)}{(1-p^2t^2-q^2z^2)} + t + c\right) \tag{13.15}$$

where c is an arbitrary constant.

Chandrasekhar and Xanthopoulos have chosen the constant of integration in (13.15) such that $\omega = 0$ when $t = 1$. The reason for this choice is associated with the convenience of aligning the coordinate directions with the shear axes on the surface $f + g = 0$ on which the two waves mutually focus each other. Accordingly we set

$$c = -1. \tag{13.16}$$

A variation of this constant is equivalent to making an Ehlers transformation as described in (12.16–18). With (13.16), ω may now be written in either of the forms

$$\begin{aligned}\omega &= -\frac{2q}{(1+p)}\frac{(1-t)(1-pt-z^2-pz^2)}{(1-p^2t^2-q^2z^2)} \\ &= \frac{2q}{p(1+p)} - \frac{2q(1-z^2)(1-pt)}{p(1-p^2t^2-q^2z^2)}.\end{aligned} \tag{13.17}$$

For purposes of simplification, it is convenient to introduce the terms

$$\begin{aligned}X &= (1-pt)^2 + q^2z^2 \\ Y &= 1-p^2t^2-q^2z^2 \ = \ p^2(1-t^2)+q^2(1-z^2).\end{aligned} \tag{13.18}$$

In view of (12.43) and (13.1), in this case we must have

$$e^{-U} = \sqrt{1-t^2}\sqrt{1-z^2} \ = \ 1-u^2-v^2. \tag{13.19}$$

The metric function M can also be obtained from (12.42) and (13.1), giving

$$e^{-M} = \frac{X}{\sqrt{1-u^2}\sqrt{1-v^2}}. \tag{13.20}$$

These expressions now complete all the metric functions contained in the line element (11.4), which may now be written as

$$\begin{aligned}\mathrm{d}s^2 = &\frac{2X\mathrm{d}u\,\mathrm{d}v}{\sqrt{1-u^2}\sqrt{1-v^2}} - \frac{Y}{X}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)^2 \\ &- \frac{4q(1-z^2)(1-pt)}{pX}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)\mathrm{d}y \\ &- \frac{(1-z^2)}{p^2XY}\Big(p^2(1-t^2)X^2 + 4q^2(1-z^2)(1-pt)^2\Big)\mathrm{d}y^2.\end{aligned} \tag{13.21}$$

After some rearrangement, this may be written in the more convenient form

$$\begin{aligned}\mathrm{d}s^2 = &\frac{X}{2}\left(\frac{\mathrm{d}t^2}{1-t^2} - \frac{\mathrm{d}z^2}{1-z^2}\right) - \frac{Y}{X}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)^2 \\ &- \frac{4q(1-z^2)(1-pt)}{pX}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)\mathrm{d}y \\ &- \frac{(1-z^2)}{p^2X}\Big(\big((1-pt)^2+q^2\big)^2 + p^2q^2(1-t^2)(1-z^2)\Big)\mathrm{d}y^2\end{aligned} \tag{13.22}$$

in which Y does not appear in a denominator.

In order to show that this is indeed the Kerr solution, we first relabel the parameters by putting

$$p = -\sqrt{M^2 - a^2}/M, \qquad q = a/M \tag{13.23}$$

which satisfies the condition $p^2 + q^2 = 1$ for arbitrary constants M and a. It is also appropriate to introduce the new coordinates r and θ instead of t and z given by

$$t = (r - M)/\sqrt{M^2 - a^2}, \qquad z = \cos\theta \tag{13.24}$$

where it may be noticed that, using the previous notation, $\lambda = \pi/2 - \theta$. The remaining coordinates may also be transformed by putting

$$\tau = -\sqrt{2}\, M\left(x - \frac{2q}{p(1+p)}\, y\right), \qquad \phi = \frac{\sqrt{2}\, M}{\sqrt{M^2 - a^2}}\, y. \tag{13.25}$$

It is also convenient to introduce the definitions

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{13.26}$$

With these substitutions the line element (13.22) may then be written in the form

$$\begin{aligned} 2M^2\mathrm{d}s^2 = {} & \left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}\tau^2 - \frac{4aMr}{\rho^2}\sin^2\theta\,\mathrm{d}\tau\mathrm{d}\phi \\ & - \left(r^2 + a^2 - \frac{2a^2Mr}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\phi^2 - \rho^2\left(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\right) \end{aligned} \tag{13.27}$$

which after rescaling $\mathrm{d}s$ is the standard form of the Kerr solution.

For this solution to describe the interaction region of colliding waves, the coordinates must satisfy the inequality $|z| < t \leq 1$. With (13.24), this implies that

$$-(M^2 - a^2)\sin^2\theta < \Delta \leq 0 \tag{13.28}$$

which is satisfied only by the region of the Kerr space-time that is inside the ergo-sphere. This is not unexpected, as this is the only region of Kerr space-time in which the Killing vectors are both space-like.

The plane on which the two waves collide is given by $t = 0$, $z = 0$, or by $r = M$, $\theta = \pi/2$ which is between the two horizons. The hypersurface on which they mutually focus each other, which is given by $f + g = 0$ or $t = 1$, is here given by $\Delta = 0$ which determines the horizons of the

Kerr solution. Thus, the curvature singularity which usually occurs when $f+g=0$, in this case, is replaced by a coordinate singularity which is normally interpreted as a horizon of the Kerr solution.

The components of the Weyl tensor inside region IV can be derived using (11.10) and have been evaluated by Chandrasekhar and Xanthopoulos. The space-time is of algebraic type D and, in particular, it may be noted that

$$\Psi_2 = \frac{1}{(1-pt-iqz)^3} = \frac{M^3}{(r-ia\cos\theta)^3}. \tag{13.29}$$

The curvature is clearly non-singular as $t\to 1$ except in the aligned case $p=1$, $q=0$ (or $a=0$). In this case only, the surface $t=1$ corresponds to a curvature singularity.

In the interaction region t is a future pointing time-like coordinate. It thus follows from (13.24) that r is also a future pointing time-like coordinate which increases from the collision at $r=M$ to the outer horizon at $r=M+\sqrt{M^2-a^2}$. In this case a possible further extension beyond the horizon would present no problems.

It may be noted, however, that if the signs of the expressions for t, p and q in (13.24) and (13.23) are changed, the Kerr metric (13.27) is still obtained but the orientation of r is altered. In this case, the collision at $r=M$ is followed by the inner horizon at $r=M-\sqrt{M^2-a^2}$ and any extension would lead to a future time-like curvature singularity at $r=0$.

The difference between these two solution has been further analysed by Hoenselaers and Ernst (1990). It is the latter case that has been described by Chandrasekhar and Xanthopoulos (1986*b*) in which there is an extension beyond the horizon at $t=1$ which contains a time-like singularity analogous to the ring singularity[1] of the Kerr solution at $r=0$, $\theta=\pi/2$. Since this singularity is time-like, it would be missed by most observers. In the other case, which has been given explicitly above, there is an analytic extension which is the asymptotically flat exterior Kerr solution. It may be pointed out, however, that in both cases the extension beyond the horizon is not unique.

When considering the question of the character of the coordinate singularity at $t=1$, Chandrasekhar and Xanthopoulos (1986*b*) have given a transformation by which it can be removed. They have found it convenient to introduce a decreasing time-like coordinate given by $s=1-u^2-v^2$. The transformation

$$\xi = s\,e^{+x/q}, \qquad \zeta = s\,e^{-x/q} \tag{13.30}$$

[1] For a description of the structure of the Kerr solution, see Hawking and Ellis (1973), or Chandrasekhar (1983).

then removes the coordinate singularity at $s = 0$, or $t = 1$. This provides an analytic extension beyond the Cauchy horizon which is another part of Kerr space-time. This extension, in one case, is followed by a time-like curvature singularity which corresponds to the source of the Kerr solution.

It may be seen that the singularity structure of this solution is very similar to that of the degenerate Ferrari–Ibañez solution described in detail in Section 10.5. In fact this solution reduces to either of the two cases of the degenerate Ferrari–Ibañez solution in the limit as $q \to 0$, which is the limit as the polarization of the approaching waves become aligned. The non-singular case occurs when $p = -1$. It may be noted that in this limit $a \to 0$, the rotation of the Kerr solution vanishes, and the solution reduces to the same part of the Schwarzschild space-time as the degenerate Ferrari–Ibañez solution.

It is now appropriate to consider the extension of the solution into the prior regions I, II and III which describe the approaching waves that give rise to this particular interaction. The metric (13.21) may easily be extended into region II simply by replacing both t and z by u. It turns out to be convenient to make the coordinate transformation

$$\tilde{x} = x + \frac{2q}{1+p}y. \tag{13.31}$$

With this, the line-element in region II takes the form

$$\mathrm{d}s^2 = \frac{2X}{\sqrt{1-u^2}}\mathrm{d}u\,\mathrm{d}v - (1-u^2)\left(X\mathrm{d}y^2 + \frac{1}{X}(\mathrm{d}\tilde{x} - 2qu\,\mathrm{d}y)^2\right) \tag{13.32}$$

where $X = 1 - 2pu + u^2$. The line-element for region III is identical to (13.32) except that v replaces u. Both these line-elements are then continuous with the Minkowski space (3.6) in region I.

The plane wave metric given by (13.32) has the single component of the curvature tensor given by

$$\Psi_4 = -(p - iq)\delta(u) - \frac{3\,(X - 2iqu)}{X^4\sqrt{X^2 + 4q^2u^2}}\frac{(1 - pu - iqu)^3}{(p + iq)^2}\Theta(u). \tag{13.33}$$

It may immediately be observed that the approaching waves have variable polarization except in the aligned limit $q \to 0$. They include an impulsive component and a step component.

As in the degenerate Ferrari–Ibañez solution, the singularity structure is thus complicated by the fact that the solution contains impulsive waves on the wave fronts $u = 0$ and $v = 0$ which form the boundaries of the interaction region. The presence of these wave components introduces

additional distribution valued singularities at the points $u = 0$, $v = 1$ and $v = 0$, $u = 1$. As described in Chapter 8, these point singularities ensure the existence of fold singularities on the surfaces $u = 1$ and $v = 1$ in regions II and III respectively.

The profiles for the approaching waves in both cases have been described in more detail by Hoenselaers and Ernst (1990). They have shown that, in the case where the Cauchy horizon corresponds to the inner Kerr horizon, the amplitude diverges towards the fold singularity which is thus a non-scalar curvature singularity. On the other hand, when the Cauchy horizon corresponds to the outer Kerr horizon, the amplitude decays towards a quasiregular fold singularity.

13.4 Other solutions

It may be noticed that the solutions presented in the previous two sections have been obtained using the generating techniques described in Sections 12.3 and 12.4 respectively. Other techniques may also be used, both in isolation and in various combinations. Since they involve the same Ernst equation, it is clear that all the solution generating techniques that have been developed for stationary axisymmetric space-times in recent years can also be applied to colliding plane wave situations.

A number of papers have recently been published which present new colliding wave solutions using some of these techniques, and it is likely that many more such papers will appear. Most of these concentrate on a description of a particular technique and how it may be adapted to the colliding wave situation. At present, however, there is less emphasis on the physical significance of the particular solution generated. Nevertheless, solutions with a number of parameters have been obtained, and general classes of solutions have been defined.

Ideally, one would like to be able to specify the particular solution corresponding to an arbitrary set of initial conditions. Such techniques are available in the axisymmetric case, and may soon be extended to the colliding wave situation. The initial attempts at the solution of this problem are mentioned in the next chapter.

So far most of the explicit solutions that have been presented have involved impulsive wave components. In the notation of (7.11), they have used $n_1 = n_2 = 2$. Much less attention has been given to smooth-fronted waves of finite duration.

Some further general aspects of cases involving impulsive components have been clarified by Ferrari (1988). In particular, it has been shown that the proper time between the collision and the subsequent singularity is

inversely proportional to the square root of the amplitudes of the impulsive components of the approaching waves. This time is also affected by their relative polarization and is a minimum when the approaching waves are colinear. This result also applies to solutions which contain a Cauchy horizon rather than a curvature singularity. In addition, Ferrari has also shown that the shock waves which accompany the impulsive waves only affect the rate at which the Weyl scalars diverge on the singularity.

Having made these general remarks, we may now briefly review the new exact solutions that have been presented. The generation techniques themselves will not be described in detail here.

First, there is the important class of solutions that has been obtained by Ferrari, Ibañez and Bruni (1987*a,b*) using the inverse scattering method developed by Belinskii and Zakharov (1978, 1979) and Carr and Verdaguer (1983). Using this method, they have obtained a two-parameter class of colliding wave solutions with non-aligned polarization. This in fact is a generalization of the soliton solution of Ferrari and Ibañez (1987*b*) described in Sections 10.4 and 10.5.

The Ferrari–Ibañez–Bruni solution has been obtained using as seed the Kasner metric. They utilize the Belinskii–Zakharov soliton technique with two real poles. The resulting metric for region IV is singular on the space-like surface $f+g=0$, except in a particular limiting case. By extending the solution back into regions II and III, it can be seen that the approaching waves must have both an impulsive component and a step component with variable polarization.

The limiting case of the Ferrari–Ibañez–Bruni solution is particularly interesting. It is of type D, and can be shown to be part of the Taub–NUT solution[2] in the Taub region where there are two space-like Killing vectors. The line element can be written in the form

$$\begin{aligned} \mathrm{d}s^2 = {} & C(1+2p\sin\psi+\sin\psi^2)(\mathrm{d}\psi^2-\mathrm{d}\lambda^2) \\ & -\left(\frac{1-\sin^2\psi}{1+2p\sin\psi+\sin^2\psi}\right)(\mathrm{d}x-2q\sin\lambda\mathrm{d}y)^2 \\ & -\cos^2\lambda(1+2p\sin\psi+\sin^2\psi)\mathrm{d}y^2 \end{aligned} \tag{13.34}$$

where p and q are constants satisfying $p^2+q^2=1$. It can immediately be seen that, when $q=0$ and $p=\pm1$, this further reduces to the two colinear degenerate Ferrari–Ibañez solutions that have already been considered in Section 10.5. The structure of this solution has been further examined by Ferrari and Ibañez (1988).

[2] Newman, Tamburino and Unti, (1963); see also Hawking and Ellis (1973) section 5.8.

Ernst, García-Díaz and Hauser (1987*a*) have shown that an Ehlers transformation (12.13) applied to the Ernst function of the Nutku–Halil solution (13.3), together with a simple coordinate transformation, yields a non-colinear generalization of the Ferrari–Ibañez soliton solution that was described in Section 10.4 but only with even integer values of the parameter a. This generalization is in fact contained in the solution of Ferrari, Ibañez and Bruni (1987*a,b*).

A further generalization extending the Ferrari–Ibañez–Bruni solution was reported by Ernst, García-Díaz and Hauser (1987*a,b*), with a full derivation given in Ernst, García-Díaz and Hauser (1988). This solution contains three arbitrary parameters. In stating this solution, it is convenient first to define the function

$$\begin{aligned} T(a,\alpha,\beta) = {} & \frac{\sqrt{1-t^2}}{2}\left[(p+p')\left(\frac{1-t}{1+t}\right)^{a/2} + (p-p')\left(\frac{1+t}{1-t}\right)^{a/2}\right] \\ & + \frac{i\sqrt{1-z^2}}{2}\left[(q+q')\left(\frac{1-z}{1+z}\right)^{a/2} + (q-q')\left(\frac{1+z}{1-z}\right)^{a/2}\right] \end{aligned} \tag{13.35}$$

where a, α and β are the arbitrary parameters, and where

$$p=\cos\alpha, \qquad q=\sin\alpha, \qquad p'=\cos\beta, \qquad q'=\sin\beta. \tag{13.36}$$

With this definition, the Ernst function for this new solution may be written as

$$Z(a,\alpha,\beta) = (1-t^2)^{a/2}(1-z^2)^{a/2}\frac{T(a+1,\beta,\alpha)}{T(a-1,\beta,\alpha)}. \tag{13.37}$$

When $a=0$, this reduces to the Nutku–Halil solution. With $|a|=1$, one obtains the Kerr–NUT solution with α being the NUT parameter. The subcase with $|a|=1$ and $\beta=0$ becomes the Schwarzschild–NUT space-time.

Transforming (13.37) using the Neugebauer–Kramer involution produces another three-parameter family of solutions with the alternative Ernst function given by

$$Z(a,\alpha,\beta) = (1-t^2)^{(1-a)/2}(1-z^2)^{(1-a)/2}\frac{T(a-2,\alpha,\beta)}{T(a,\alpha,\beta)}. \tag{13.38}$$

With $a=0$, this reduces to the Chandrasekhar–Xanthopoulos solution.

Ernst, García-Díaz and Hauser (1987*b*) have also found a simpler realization $G_0(\Sigma_1)$ of the Geroch group of transformations relating colliding wave solutions. With this , they have obtained a generalization of the

above solution in the case when $a = 2$. A further generalization of the $a = 3$ solution is also obtained using a Neugebauer–Kramer involution.

It was pointed out by Ernst (1986) that the Nutku–Halil solution can be derived from the isotropic ($n = 0$) Kasner metric by applying a double-Harrison (Bäcklund) transformation. The Kerr space-time of the Chandrasekhar–Xanthopoulos solution can also be derived from Minkowski space using a double-Harrison transformation. It therefore seemed reasonable that other colliding wave solutions may be generated by applying double-Harrison transformations to other Kasner metrics. This procedure has indeed been successful. It was used by Ernst, García-Díaz and Hauser (1988) to determine the solution described above in (13.35) to (13.38).

The effect of a double-Harrison transformation can be deduced by any number of competing techniques. Naturally, however, Ernst, García-Díaz and Hauser preferred to use the so-called Hauser–Ernst homogeneous Hilbert problem approach (see also Ernst 1988).

Finally, it may be mentioned that Hassan, Feinstein and Manko (1990) have recently obtained another particular solution describing the non-colinear case using the generating technique of Gutsunaev and Manko (1988) as described here in (12.56) and (15.57). Further solutions could easily be obtained using this method.

14

THE INITIAL-VALUE PROBLEM

The approach to finding exact solutions that has been taken so far has involved initially solving the field equations in the interaction region IV and then investigating the conditions under which these solutions can be considered as the result of collisions of plane waves. In this way, resulting solutions are found first and the initial conditions are obtained subsequently. In this chapter it is appropriate to return to the original problem of specifying the initial data and then attempting to find the solution that determines the subsequent development.

14.1 The initial data

The problem that is under consideration is the collision of two plane waves in a flat background. It is assumed that the two approaching waves are both known, and it is required to find the exact solution which describes the interaction following the collision. This problem has been formulated in earlier chapters. It has been found convenient to divide space-time into the four regions as illustrated in Figure 3.1. It has also been demonstrated that the metric in the interaction region IV may be taken in the form of the Szekeres line element (6.20) which involves the four functions $U(u,v)$, $V(u,v)$, $W(u,v)$ and $M(u,v)$ satisfying equations (6.22). One of these equations can be integrated to give

$$e^{-U} = f(u) + g(v). \tag{14.1}$$

In this approach, it is assumed that the initial conditions are determined by the functions $f(u)$, $V(u,0)$, $W(u,0)$ and $M(u,0)$ which represents the wave in region II as it reaches the boundary $v = 0$, and by $g(v)$, $V(0,v)$, $W(0,v)$ and $M(0,v)$ which represents the wave in region III as it reaches the boundary $u = 0$. This is now a typical case of the characteristic initial-value problem.

In practice, however, things are a little more complicated since it is normal to make use of the transformation (6.7) to put $M = 0$ in these initial regions. In this case the approaching waves are each described by three functions either of u or of v satisfying a single equation which is either (6.22*c*) or (6.22*b*).

The difficulty arises because, as described in previous chapters, it is convenient to use f and g, or transformations of them, as coordinates in

the interaction region. Accordingly, it is therefore more convenient to use the transformation (6.7) in the initial regions to put

$$f = \tfrac{1}{2} - u^{n_1}\Theta(u), \qquad g = \tfrac{1}{2} - v^{n_2}\Theta(v). \tag{14.2}$$

where the constants $n_1 \geq 2$ and $n_2 \geq 2$ are essential in order to retain the continuity properties of the functions f and g across the boundaries $v = 0$ and $u = 0$. These constants feature prominently in the junction conditions as described in Section 7.2. They are determined by the character of the wavefronts of the approaching waves.

In this case, the data specifying the initial waves is now given by $V(u, 0)$, $W(u, 0)$ and $M(u, 0)$ on $v = 0$, and by $V(0, v)$, $W(0, v)$ and $M(0, v)$ on $u = 0$. In addition, since the function M is essentially determined up to a removable constant for any V and W by equations (6.22*b,c*), the initial data is effectively described by specifying only the metric functions V and W on the boundaries $v = 0$ and $u = 0$. It is also appropriate to re-express these as functions of f and g.

The remaining problem is now to integrate the main field equations to determine the functions V and W in the interaction region subject to their specification on the initial null boundaries.

14.2 The colinear case

For the case when the approaching waves have aligned linear polarization it is possible to put $W = 0$ everywhere, and the main field equation for V may be expressed in terms of different coordinates in any of the forms (9.3), (10.2), (10.11), (10.60) or (10.71). At this point we may take f and g as coordinates and consider the equation in the form (10.2) which may be rewritten as

$$L[V] = V_{fg} + \frac{1}{2(f+g)}V_f + \frac{1}{2(f+g)}V_g = 0. \tag{14.3}$$

This equation must now be solved for $V(f, g)$ with initial data defining the approaching waves given by $V(f, \frac{1}{2})$, and $V(\frac{1}{2}, g)$. It is also possible to scale the approaching waves such that $V = 0$ at the point of collision $u = 0$, $v = 0$ so that $V(\frac{1}{2}, \frac{1}{2}) = 0$.

Equation (14.3) is an Euler–Poisson–Darboux equation whose solution may be expressed as a line integral. As originally pointed out in this context by Szekeres (1972) and later repeated by Yurtsever (1988*c*), since (14.3) is a linear hyperbolic equation, it may also be solved explicitly using Riemann's method. According to this method, we consider the adjoint equation

$$\tilde{L}[R] = R_{fg} - \left(\frac{R}{2(f+g)}\right)_{,f} - \left(\frac{R}{2(f+g)}\right)_{,g} = 0 \tag{14.4}$$

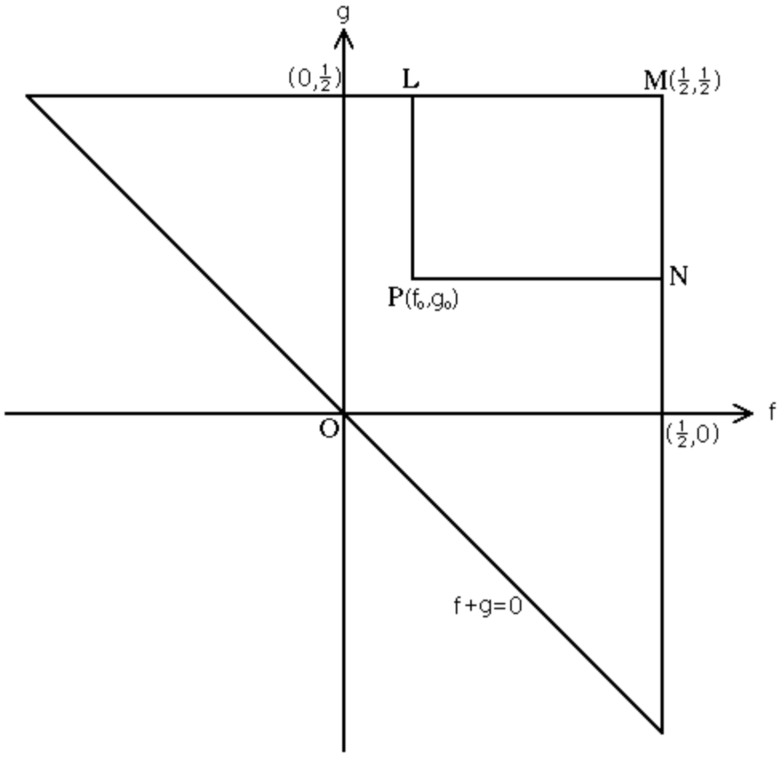

Figure 14.1 The interaction region IV is represented in *f,g* coordinates by the region inside the triangle shown. The sides $g = 1/2$ and $f = 1/2$ are the II–IV and III–IV boundaries respectively, and the focusing singularity occurs on the line $f + g = 0$. The solution for *V(f,g)* may be obtained at any point P by integrating round the rectangle PNML.

where R is a Riemann–Green function satisfying the boundary conditions

$$\begin{aligned} 2R_f - \frac{R}{(f+g)} &= 0 \qquad \text{at} \qquad g = g_\mathrm{o} \\ 2R_g - \frac{R}{(f+g)} &= 0 \qquad \text{at} \qquad f = f_\mathrm{o} \\ R(f_\mathrm{o}, g_\mathrm{o}) &= 1. \end{aligned} \tag{14.5}$$

By using Green's theorem the integral of $RL[V] - V\tilde{L}[R]$ over the rectangle PNML indicated in Figure 14.1 can be related to its line integral around the boundary. In this way it can be shown that, for any arbitrary point $(f_\mathrm{o}, g_\mathrm{o})$ within the interaction region, the function V is given by

$$V(f_\mathrm{o}, g_\mathrm{o}) = \int_{ML} R\left[V_f + \frac{V}{2(\frac{1}{2}+f)}\right]\mathrm{d}f + \int_{MN} R\left[V_g + \frac{V}{2(\frac{1}{2}+g)}\right]\mathrm{d}g \tag{14.6}$$

provided that R satisfies (14.4) and (14.5) and that $V(\frac{1}{2},\frac{1}{2})=0$.

A specific Riemann–Green function which satisfies (14.4) and (14.5) is given by

$$R(f,g;f_o,g_o)=\sqrt{\frac{f+g}{f_o+g_o}}P_{-1/2}\left(1+\frac{2(f-f_o)(g-g_o)}{(f+g)(f_o+g_o}\right) \tag{14.7}$$

where $P_{-1/2}$ is the Legendre function of order $-\frac{1}{2}$. By substituting this and the initial expressions for V on the boundaries ML and MN into (14.6), an explicit integral for V in the interaction region is obtained.

This approach of using Riemann's method has been generalized by Xanthopoulos (1986) to include the collision of null fluids coupled with plane gravitational waves. This particular situation will be discussed later in Section 20.2. At this point, we may simply note that the method involved is essentially the same.

Although the method described above does give an explicit integral expression for V, in practice it is extremely difficult to evaluate this integral for arbitrary initial data. Thus, it does not really provide a viable method for generating analytic solutions for any given analytic description of the approaching waves.

Another problem with the method of Riemann is that it is not practically possible to generalize it to the case for non-colinear collisions. The method only applies to linear hyperbolic equations and, if the polarization of the approaching waves is not aligned, the field equations describing the resulting interaction are non-linear. In view of this, Hauser and Ernst (1989*a,b*, 1990) have developed an alternative method which can be generalized to the non-colinear situation.

The method employed by Hauser and Ernst (1989*a*) is first to find a one-parameter family of basic solutions by means of a suitable separation of variables and then to express the the final solution as a linear superposition of these basic solutions. In fact their approach is based on the solution (10.8), which may be re-expressed in the form

$$\begin{aligned}V=&\int_{1/2}^{f(u)}\frac{A(\sigma)\,\mathrm{d}\sigma}{\sqrt{\sigma-f(u)}\sqrt{\sigma+g(v)}}\\&+\int_{1/2}^{g(v)}\frac{B(\sigma)\,\mathrm{d}\sigma}{\sqrt{\sigma+f(u)}\sqrt{\sigma-g(v)}}\end{aligned} \tag{14.8}$$

for suitable functions $A(\sigma)$ and $B(\sigma)$ which must depend on the initial data.

An interesting feature of the method is that the functions $A(\sigma)$ and $B(\sigma)$ are obtained in terms of the prescribed initial data by solving generalized versions of Abel's integral equation for the tautochrone problem of classical particle mechanics. Generalized versions of Abel's integral equation are required because $f'(u)$ and $g'(v)$ are zero on the boundaries $u = 0$ and $v = 0$ respectively.

Full details of the method have been described by Hauser and Ernst (1989*a*). However, the techniques are still difficult to use in practice for arbitrary initial data.

Hauser and Ernst (1989*b*) have also reformulated the initial-value problem for the colinear case as a Hilbert problem in a complex plane in two different ways. They have presented solutions of both forms of the Hilbert problem and shown that each of these agrees with the solution that is obtained by the method referred to above (Hauser and Ernst, 1989*a*).

14.3 The non-colinear case

Hauser and Ernst (1989*a,b*, 1990) have developed the techniques mentioned at the end of the previous section to deal specifically with the case when the polarization of the approaching waves is not aligned.

The initial data have already been described in Section 14.1. It was also pointed out that, for the colinear case, it was convenient to rescale the approaching waves such that $V = 0$ at the point of collision. For the non-colinear case, Hauser and Ernst have found it convenient to work with the complex function Z defined by (11.5) and to use the transformation (12.20) which consists of a rotation and rescaling to put $Z = 1$ at the point of collision when $u = 0$ and $v = 0$.

For the colinear case the main field equation (14.3) is linear. However, for the non-colinear case this is replaced by two non-linear equations, or by the complex Ernst equation. The non-linearity of this equation introduces considerable difficulty into any attempt at solving the general problem.

Hauser and Ernst (1990) have developed a method for considering the initial-value problem for general colliding plane gravitational waves. They have achieved this by replacing the usual initial-value problem in terms of the Ernst equation by an equivalent 2×2 matrix homogeneous Hilbert problem in the complex plane. In the case when the polarization of the approaching waves is colinear, this approach reduces to the relatively simple one-dimensional Hilbert problem that was previously considered (Hauser and Ernst, 1989*b*). A detailed description of the method developed, however, is beyond the scope of this book and interested readers are directed to the original papers.

In spite of these advances, there is strong evidence that the general solution of the initial-value problem for the non-colinear case is not expressible in a finite closed form. In addition, practical methods for solving the matrix homogeneous Hilbert problem for particular non-colinear cases still need to be developed.

Clearly this is an area where considerable further work is required.

15

COLLIDING ELECTROMAGNETIC WAVES: THE BELL–SZEKERES SOLUTION

In an important paper, Bell and Szekeres (1974) gave an exact solution which describes the collision and subsequent interaction of two electromagnetic plane waves. This solution, which appears to be remarkably simple, will be described in this chapter. Other exact solutions describing colliding electromagnetic waves will be described in Chapters 16 and 17, after more powerful techniques have been developed.

15.1 The Bell–Szekeres solution

Bell and Szekeres have considered a very simple situation involving a collision of two step electromagnetic waves. Prior to the collision, the waves may be described by the familiar line element (4.15). In regions I and II, this takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a^2\Theta(u)(X^2 + Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{15.1}$$

and the step wave is given by $\Phi_{22} = a^2\Theta(u)$. The opposing wave with $\Phi_{00} = b^2\Theta(v)$ may be described in an identical way with the null coordinate u replaced by v, and a different space-like coordinate replacing r.

The two electromagnetic field components in the initial regions I, II and III are taken to be

$$\Phi_2 = a\Theta(u), \qquad \text{and} \qquad \Phi_0 = b\Theta(v). \tag{15.2}$$

The fact that both of these expressions are real simultaneously indicates that the polarization of the two electromagnetic waves is aligned. In addition, they have identical step profiles. The relative amplitudes a and b may, of course, both be equated to unity when convenient. The components of the Weyl tensor are initially all zero.

By this stage, we are familiar with the fact that it is appropriate to transform the metrics in all regions into Rosen form. Accordingly, the line elements in the initial regions may be taken in the forms of (4.19) and (4.20), namely

$$\begin{array}{lll}
\text{I} \ (u<0, v<0) & \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 & \\
\text{II} \ (u\geq 0, v<0) & \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 au(\mathrm{d}x^2 + \mathrm{d}y^2) & (15.3)\\
\text{III} \ (u<0, v\geq 0) & \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 bv(\mathrm{d}x^2 + \mathrm{d}y^2). &
\end{array}$$

The initial-value problem is now well set, and it remains to find the unique solution in the interaction region.

The field equations for colliding electromagnetic waves have already been obtained in Chapter 6. In terms of the metric functions of the Szekeres line element (6.20), the gravitational field equations for the interaction region take the form (6.22), and Maxwell's equations are given by (6.21). The appropriate boundary conditions for this situation have been described in Chapter 7.

As usual, it may be noted that (6.22a) can immediately be integrated to give

$$e^{-U} = f(u) + g(v) \tag{15.4}$$

where f and g are arbitrary decreasing functions in the interaction region. From the junction conditions it can be seen that, in this case, these functions necessarily take the forms:

$$f = \tfrac{1}{2} - \sin^2 au, \qquad g = \tfrac{1}{2} - \sin^2 bv. \tag{15.5}$$

The solution of the complete set of equations (6.21,22) given by Bell and Szekeres is

$$\begin{gathered} U = -\log\cos(au - bv) - \log\cos(au + bv) \\ V = \log\cos(au - bv) - \log\cos(au + bv) \\ W = 0, \qquad M = 0, \qquad \Phi_2 = a, \qquad \Phi_0 = b. \end{gathered} \tag{15.6}$$

It can be seen that these functions satisfy the required O'Brien–Synge boundary conditions. Thus the line element in the interaction region IV is

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2(au - bv)\mathrm{d}x^2 - \cos^2(au + bv)\mathrm{d}y^2. \tag{15.7}$$

This line element is in fact one form of the Bertotti–Robinson solution (Bertotti 1959, Robinson 1959), which is known to be conformally flat. The coordinate transformation

$$\begin{aligned} t + r &= \coth\left(\frac{1}{2}\mathrm{sech}^{-1}\cos(au + bv) - \frac{y}{2q}\right) \\ t - r &= -\tanh\left(\frac{1}{2}\mathrm{sech}^{-1}\cos(au + bv) + \frac{y}{2q}\right) \\ \theta &= au - bv + \frac{\pi}{2}, \qquad\qquad \phi = \frac{x}{2} \end{aligned} \tag{15.8}$$

transforms the line element (15.7) into the more familiar form

$$\mathrm{d}s^2 = \frac{q^2}{r^2}(\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2). \tag{15.9}$$

Halilsoy (1987) has considered an application of the transformation $u \to u'(u)$, $v \to v'(v)$ to the above solution. However, this does not lead to any new results that are of physical significance.

15.2 The structure of the solution

The above solution describes the collision of two step electromagnetic waves whose polarization vectors are aligned. This follows from the fact that Φ_{02} is real, and $W = 0$.

As has been pointed out above, the solution inside the interaction region IV is conformally flat. However, as Bell and Szekeres have shown, there are necessarily discontinuities in the derivatives of the metric function V across the initial boundaries of this region. These manifest themselves as the impulsive gravitational waves

$$\Psi_4 = -a \tan bv\, \delta(u)\Theta(v), \qquad \Psi_0 = -b \tan au\, \delta(v)\Theta(u) \tag{15.10}$$

which may be considered to be generated by the collision.

It is in fact a general feature of colliding electromagnetic plane waves that gravitational waves are always generated by the collision. This can easily be demonstrated by considering the field equations (6.22*d,e*). In the interaction region, Φ_2 and Φ_0 are necessarily both non-zero. It follows that V and W can not both remain constant. Thus, the congruences tangent to both waves must begin to shear, and it can be seen from (6.23) that the components of the Weyl tensor Ψ_0 and Ψ_4, and possibly Ψ_2, will necessarily appear. In this case, the discontinuity in the electromagnetic components causes impulsive gravitational waves to be generated. It can also be shown that smooth-fronted electromagnetic waves would generate smooth-fronted or step gravitational waves that wouls persist through the interaction region.

It may also be observed from (15.10) that difficulties occur at the points $u = \pi/2a$, $v = 0$ and $v = \pi/2b$, $u = 0$. At first sight, it would appear that curvature singularities occur at these points (Matzner and Tipler, 1984). However, curvature tensors of this type can be interpreted as distributions (see Geroch and Traschen, 1987) and, in this case, the standard definition of a curvature singularity in terms of unboundedness in a parallely propagated frame is inapplicable.

The global structure of this solution has been analysed in detail by Clarke and Hayward (1989). They have confirmed that, in regions II and III, the surfaces on which $f = -1/2$ and $g = -1/2$ respectively behave as 'fold' singularities similar to those of the Khan–Penrose solution described in Section 8.2.

Initially, the most surprising feature of the Bell–Szekeres solution was that a space-like curvature singularity does not occur in region IV. In this case, the surface on which $f+g=0$, which is here given by $au+bv=\pi/2$, turns out to be merely a coordinate singularity since the curvature tensor is clearly bounded. Bell and Szekeres have demonstrated how this singularity can be removed by considering the following coordinate transformation:

$$\begin{aligned}\tilde{T} &= -\cos(au+bv)\cosh cy\\ \tilde{Z} &= \cos(au+bv)\sinh cy\\ \tilde{X} &= \cos(au-bv)\cos cx\\ \tilde{Y} &= \cos(au-bv)\sin cx\end{aligned} \tag{15.11}$$

where $c=\sqrt{2ab}$. With this, the line element (15.6) becomes

$$\begin{aligned}\mathrm{d}s^2 = &\left(\frac{(1+\tilde{Z}^2)\mathrm{d}\tilde{T}^2 - 2\tilde{T}\tilde{Z}\mathrm{d}\tilde{T}\mathrm{d}\tilde{Z} - (1-\tilde{T}^2)\mathrm{d}\tilde{Z}^2}{c^2(1-\tilde{T}^2+\tilde{Z}^2)}\right)\\ &-\left(\frac{(1-\tilde{Y}^2)\mathrm{d}\tilde{X}^2 + 2\tilde{X}\tilde{Y}\mathrm{d}\tilde{X}\mathrm{d}\tilde{Y} + (1-\tilde{X}^2)\mathrm{d}\tilde{Y}^2}{c^2(1-\tilde{X}^2-\tilde{Y}^2)}\right)\end{aligned} \tag{15.12}$$

which is clearly regular when $\tilde{T}=\tilde{Z}=0$.

In order to retain the one-to-one correspondence, it is appropriate initially to restrict the Bell–Szekeres coordinate x to the range $-\pi < cx \leq \pi$. The coordinates $\tilde{X}$ and $\tilde{Y}$ are restricted by $\tilde{X}^2+\tilde{Y}^2<1$, where the boundary $\tilde{X}^2+\tilde{Y}^2=1$ contains the space-like plane on which the collision occurs.

It has been shown by Clarke and Hayward (1989) that region IV is part of a space-time that is regular apart from two covering-space singularities that occur when $au-bv=\pm\pi/2$. These are points on the boundary of region IV where it joins with the limits of regions II and III. At these points, which are on the polar axis $\theta=0$ and $\theta=\pi$ in the Bertotti–Robinson form of the metric (15.9), $\tilde{X}=\tilde{Y}=0$.

With the above transformation, each interval of the coordinate x of length $2\pi/c$ corresponds to a circular region in the $\tilde{X},\tilde{Y}$ plane with the origin removed and cut along the line $\tilde{Y}=0$, $\tilde{X}<0$. Neighbouring regions of x are joined together along these cuts to form a surface which continuously winds around the origin. The origin of the $\tilde{X},\tilde{Y}$ plane thus corresponds to a singularity of a quasiregular covering-space type.

It is also sometimes convenient (Clarke and Hayward, 1989) to make the further coordinate transformation

$$\begin{aligned}\rho &= \sinh^{-1}\tilde{Z}\\ \chi &= \sin^{-1}\left(\frac{\tilde{T}}{\sqrt{1+\tilde{Z}^2}}\right)\\ \tilde{\theta} &= \cos^{-1}\tilde{Y}\\ \tilde{\phi} &= \sin^{-1}\left(\frac{\tilde{X}}{\sqrt{1-\tilde{Y}^2}}\right).\end{aligned} \tag{15.13}$$

With this, the line element (15.12) takes the alternative form of the Bertotti–Robinson line element

$$\mathrm{d}s^2 = \frac{1}{c^2}\left(\cosh^2\rho\,\mathrm{d}\chi^2 - \mathrm{d}\rho^2 - \mathrm{d}\tilde{\theta}^2 - \sin^2\tilde{\theta}\,\mathrm{d}\tilde{\phi}^2\right). \tag{15.14}$$

Region IV is now described by the line element (15.14) with the coordinate ranges given by

$$\begin{aligned}-\infty < \tilde{\phi} < \infty\\ -\infty < \rho < \infty\\ -\pi/2 < \chi < -\sin^{-1}|\tanh\rho|\\ \cos^{-1}(\cos\chi\cosh\rho) < \tilde{\theta} < \cos^{-1}(-\cos\chi\cosh\rho).\end{aligned} \tag{15.15}$$

This is joined to regions II and III across the surfaces

$$\cos\tilde{\theta} = \cos\chi\cosh\rho \qquad \text{and} \qquad \cos\tilde{\theta} = -\cos\chi\cosh\rho. \tag{15.16}$$

Clarke and Hayward (1989) have also shown that the Bell–Szekeres solution is extendable to the future through the surface $au + bv = \pi/2$, but the extension is not unique because of the presence of the singularities. They have further suggested two possible extensions which will be described in the next section.

It is also somewhat remarkable that the global structure of the Bell–Szekeres solution, as described by Clarke and Hayward (1989), is very similar to that of the collision of an impulsive gravitational wave with a null shell of matter, as described by Babala (1987). This will be discussed in Section 19.3.

Chandrasekhar and Xanthopoulos (1988) have conducted a thorough perturbation analysis of this particular solution. Throughout the interaction region subsequent to the collision they have obtained a complete set

of bounded normal modes that are expressed in terms of spin-weighted spherical harmonics. These modes exhibit a behaviour of ever-increasing frequency as the Cauchy horizon is approached.

Chandrasekhar and Xanthopoulos (1988) have also considered perturbations in the initial regions which contain the approaching waves. For the approaching wave in region II which propagates along the null line $u = \text{const}$, $v \leq 0$, they have shown that v-independent perturbations are not permitted to all orders, while all the v-dependent perturbations exhibit strong divergences along the 'fold' singularity. Unfortunately, these perturbations can not be joined continuously to those of the interaction region. It is therefore not possible to consider the effects of these perturbations subsequent to the collision.

15.3 Extensions of the solution

In the Bell–Szekeres solution, it has been shown that the hypersurface $f + g = 0$ on which the opposing waves mutually focus each other is a Cauchy horizon rather than a curvature singularity. It is therefore appropriate to consider possible extensions to this solution through the horizon, even though any such extension will not be unique. Two possible extensions have been given by Clarke and Hayward (1989).

One natural extension of region IV is simply the future extension of the region as described by the line element (15.14). This is achieved by extending the coordinate range to

$$\begin{gathered} -\infty < \tilde{\phi}, \rho < \infty, \qquad -\pi/2 < \chi < \infty \\ \cos^{-1}\lambda < \tilde{\theta} < \cos^{-1}(-\lambda) \end{gathered} \tag{15.17}$$

where $\lambda = \min(1, \cos\chi\cos\rho)$. In this extension, the waves cross and focus. The subsequent space-time is unchanging.

Another equally natural extension of region IV is obtained by noting that the line element (15.14) admits a reflection symmetry $\chi \to -\chi$ about the hyperplane $\chi = 0$ (or $\tilde{T} \to -\tilde{T}$ about $\tilde{T} = 0$). Thus, there is an extension in which region IV is extended as far as $\chi = 0$ and is then followed by the time reverse of the entire solution. In this extension, the waves cross, mutually focus each other, re-expand and then separate leaving Minkowski space between them to the future. This situation is illustrated in Figure 15.1 in terms of the original null coordinates and the line elements (15.3) and (15.7).

Other apparent extensions of the Bell–Szekeres solution have been suggested by Gürses and Halilsoy (1982) in which the approaching electromagnetic waves consist of a sequence of steps. The step lengths, however, have to be double the distance to the fold singularities. Thus, these

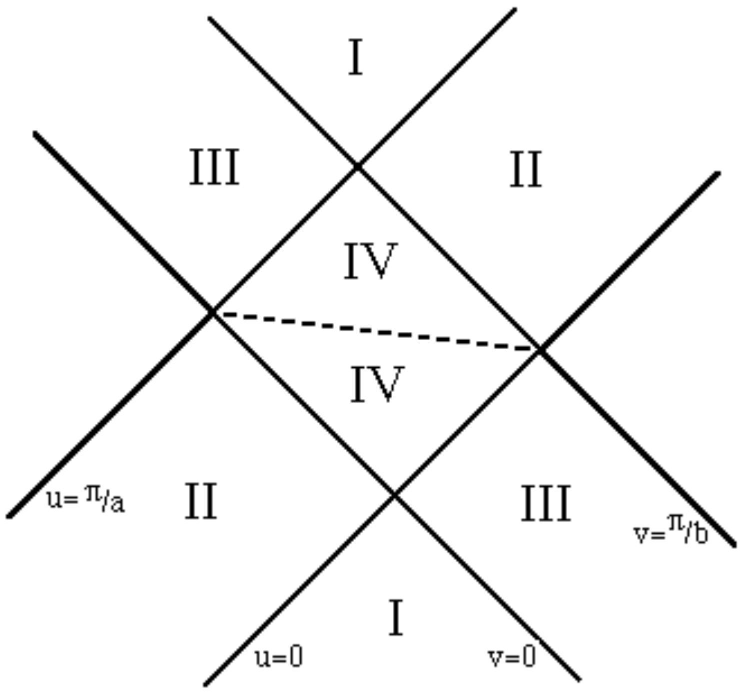

Figure 15.1 An extension of the Bell–Szekeres space-time. Regions marked I are flat. Regions marked II and III contain electromagnetic waves with $\Phi_2 = a$, and $\Phi_0 = b$ respectively, having metrics given by (15.3). There are impulsive gravitational waves along the boundaries of region IV, which has metric (15.7).

solutions necessarily extend the space-time through the covering space singularities and can not, therefore, be considered as appropriate solutions for colliding plane waves. In all these solutions the space-time can only be extended uniquely up to the fold singularities in regions II and III and the surface in region IV on which $au + bv = \pi/2$. It must therefore be concluded that the Gürses–Halilsoy extensions are not physically significant.

15.4 A non-colinear collision

In the Bell–Szekeres solution, the polarization of the approaching electromagnetic waves is aligned. It is appropriate now to consider how this solution may be generalized to include the case when the polarization is not aligned.

In the approaching waves a rotation of the polarization is expressed by a rotation of the components Φ_2 or Φ_0 in the complex plane. The metrics in both regions are unaltered, since a null electromagnetic field

is defined by the metric only up to a constant duality rotation. Thus, to extend the Bell–Szekeres solution, the same initial conditions are given except that the component Φ_{02} becomes complex and, from (6.22e), W must become non-zero in the interaction region.

A solution presented by Griffiths (1985) satisfying this condition may be written in the form

$$\begin{aligned}
e^{-U} &= \tfrac{1}{2}\cos 2au + \tfrac{1}{2}\cos 2bv \\
e^{2V} &= \frac{(1+\cos 2au\cos 2bv + \sin 2au \sin 2bv \cos\alpha)}{(1+\cos 2au\cos 2bv - \sin 2au \sin 2bv \cos\alpha)} \\
\sinh W &= \frac{\sin 2au \sin 2bv \sin\alpha}{(\cos 2au + \cos 2bv)} \\
M = 0, &\qquad \Phi_2 = ae^{i\theta}, \qquad \Phi_0 = ae^{-i\theta}
\end{aligned} \tag{15.18}$$

where

$$\theta = \tfrac{1}{2}\cos^{-1}(\cos\alpha\,\mathrm{sech}W). \tag{15.19}$$

The relative polarization angle of the two waves prior to the collision is given by α. It may be noticed that the expression for U here is equivalent to those of (15.4,5) and (15.6). The common multiple 2 has been inserted so that this solution can be more directly related to the Bell–Szekeres solution to which it reduces when $\alpha = 0$.

It can be shown that this solution has exactly the same properties as the aligned Bell–Szekeres solution. It is conformally flat everywhere except at the boundaries of the interaction region where there are impulsive gravitational waves that may be considered to be generated by the collision. These are given by

$$\begin{aligned}
\Psi_4 &= -a\tan bv\, e^{i\alpha}\delta(u)\Theta(v) \\
\Psi_0 &= -b\tan au\, e^{-i\alpha}\delta(v)\Theta(u).
\end{aligned} \tag{15.20}$$

According to a theorem of Tariq and Tupper (1974), this new solution must also be the Bertotti–Robinson space-time in a different coordinate system. In fact it can be obtained from the Bell–Szekeres line element (15.7) by the simple rotation

$$\begin{aligned}
x &= \cos\frac{\alpha}{2}x' + \sin\frac{\alpha}{2}y' \\
y &= -\sin\frac{\alpha}{2}x' + \cos\frac{\alpha}{2}y'.
\end{aligned} \tag{15.21}$$

Thus, the gravitational field in this case is identical to that of the Bell–Szekeres solution, as may have been expected. In the collision of plane

electromagnetic waves with non-aligned polarization, it must therefore be concluded that the dynamics remains unaltered. It is only the form of the metric in the interaction region that needs to be modified to that given in this section.

16

ERNST'S EQUATION FOR COLLIDING ELECTROMAGNETIC WAVES

The convenience of exploiting the remarkable analogy between stationary axi-symmetric space-times and colliding plane wave solutions has been thoroughly demonstrated in previous chapters. It has enabled many of the well established solution-generating techniques to be used to obtain new solutions for colliding gravitational waves, and opened up the possibility of finding solutions for arbitrary initial data. The methods used are based on a study of Ernst's equation, which can also be extended (Ernst, 1968*b*) to include electromagnetic fields.

The extension of the Ernst approach to colliding electromagnetic plane waves was first formulated by Chandrasekhar and Xanthopoulos (1985*a*). This approach is described in this chapter, though from a slightly different starting point.

16.1 The field equations

The derivation of Ernst's equations given here is a fairly lengthy one. The purpose is to derive them from the basic field equations given previously in (6.21) and (6.22). We therefore start here with Maxwell's equations in the form (6.21), and make the substitutions

$$\begin{aligned} P_0 &= e^{-U/2}e^{V/2}\sqrt{1+i\sinh W}\;\Phi_0^\circ \\ P_2 &= e^{-U/2}e^{V/2}\sqrt{1-i\sinh W}\;\Phi_2^\circ. \end{aligned} \tag{16.1}$$

With this Maxwell's equations take the form

$$\begin{aligned} P_{2,v} &= \tfrac{1}{2}(1+i\sinh W)(V_v - iW_v\operatorname{sech} W)P_2 \\ &\quad - \tfrac{1}{2}(1-i\sinh W)(V_u + iW_u\operatorname{sech} W)P_0 \\ P_{0,u} &= \tfrac{1}{2}(1-i\sinh W)(V_u + iW_u\operatorname{sech} W)P_0 \\ &\quad - \tfrac{1}{2}(1+i\sinh W)(V_v - iW_v\operatorname{sech} W)P_2. \end{aligned} \tag{16.2}$$

From this, it is clear that $P_{0,u} = -P_{2,v}$, which implies that there exists a complex potential function $H(u,v)$ such that

$$P_2 = H_u, \qquad P_0 = -H_v. \tag{16.3}$$

Thus

$$\Phi_0^\circ = -\frac{e^{U/2}e^{-V/2}}{\sqrt{1+i\sinh W}}\,H_v, \qquad \Phi_2^\circ = \frac{e^{U/2}e^{-V/2}}{\sqrt{1-i\sinh W}}\,H_u \tag{16.4}$$

and Maxwell's equations reduce to the single equation

$$\begin{aligned} H_{uv} &= \tfrac{1}{2}(1+i\sinh W)(V_v - iW_v\,\mathrm{sech}\,W)H_u \\ &\quad + \tfrac{1}{2}(1-i\sinh W)(V_u + iW_u\,\mathrm{sech}\,W)H_v. \end{aligned} \tag{16.5}$$

It is now convenient to change to the metric functions χ and ω using (11.2) and (11.3), so that the line element takes the form (11.4). In terms of these functions, the main field equations (6.22d,e) now become

$$\begin{aligned} &2\chi_{uv} - U_u\chi_v - U_v\chi_u - \frac{2}{\chi}(\chi_u\chi_v - \omega_u\omega_v) - 2\chi^2e^U(H_u\bar{H}_v + \bar{H}_uH_v) = 0 \\ &2\omega_{uv} - U_u\omega_v - U_v\omega_u - \frac{2}{\chi}(\chi_u\omega_v + \chi_v\omega_u) - 2i\chi^2e^U(H_u\bar{H}_v - \bar{H}_uH_v) = 0 \end{aligned} \tag{16.6}$$

and Maxwell's equations imply that

$$2\chi H_{uv} + (\chi_v + i\omega_v)H_u + (\chi_u - i\omega_u)H_v = 0 \tag{16.7}$$

where

$$\Phi_0^\circ = -\sqrt{\frac{\chi\,(\chi - i\omega)}{(f+g)\sqrt{\chi^2+\omega^2}}}\,H_v, \qquad \Phi_2^\circ = \sqrt{\frac{\chi\,(\chi + i\omega)}{(f+g)\sqrt{\chi^2+\omega^2}}}\,H_u. \tag{16.8}$$

By way of a digression it may be recalled that, in Chapter 11, it was found convenient to introduce the complex function $Z_\mathrm{o} = \chi + i\omega$. With this, the equations (16.6) may be written as the single complex equation

$$(Z_\mathrm{o}+\bar{Z}_\mathrm{o})(2Z_{\mathrm{o}uv} - U_uZ_{\mathrm{o}v} - U_vZ_{\mathrm{o}u}) - 4Z_{\mathrm{o}u}Z_{\mathrm{o}v} - (Z_\mathrm{o}+\bar{Z}_\mathrm{o})^3e^U\bar{H}_uH_v = 0 \tag{16.9}$$

and (16.7) becomes

$$(Z_\mathrm{o}+\bar{Z}_\mathrm{o})H_{uv} - Z_{\mathrm{o}v}H_u - \bar{Z}_{\mathrm{o}u}H_v = 0. \tag{16.10}$$

However, this approach does not seem to lead to anything useful.

It is in fact more convenient at this point to adopt the approach of Chandrasekhar and Ferrari (1984), and to change to the coordinates t

and z using (10.9) to (10.12). In this case, the main field equations (16.6) can be rewritten as

$$\begin{aligned}\left(\frac{(1-t^2)}{\chi}\chi_t\right)_{,t} - \left(\frac{(1-z^2)}{\chi}\chi_z\right)_{,z} + \frac{(1-t^2)}{\chi^2}\omega_t^2 - \frac{(1-z^2)}{\chi^2}\omega_z^2 \\ - 2\chi\left(\frac{\sqrt{1-t^2}}{\sqrt{1-z^2}}H_t\bar{H}_t - \frac{\sqrt{1-z^2}}{\sqrt{1-t^2}}H_z\bar{H}_z\right) = 0 \qquad (16.11)\\ \left(\frac{(1-t^2)}{\chi^2}\omega_t\right)_{,t} - \left(\frac{(1-z^2)}{\chi^2}\omega_z\right)_{,z} - 2i(H_z\bar{H}_t - H_t\bar{H}_z) = 0.\end{aligned}$$

It may be observed that the second of these equations may be written in the form

$$\left(\frac{(1-t^2)}{\chi^2}\omega_t + i(H\bar{H}_z - \bar{H}H_z)\right)_{,t} - \left(\frac{(1-z^2)}{\chi^2}\omega_z + i(H\bar{H}_t - \bar{H}H_t)\right)_{,z} = 0 \qquad (16.12)$$

from which it is clear that, as in (12.34), there exists a real potential function Φ such that

$$\begin{aligned}\Phi_z &= \frac{(1-t^2)}{\chi^2}\,\omega_t + i(H\bar{H}_z - \bar{H}H_z)\\ \Phi_t &= \frac{(1-z^2)}{\chi^2}\,\omega_z + i(H\bar{H}_t - \bar{H}H_t).\end{aligned} \qquad (16.13)$$

At this point, it is convenient to re-introduce the function Ψ of (12.36) defined by

$$\Psi = \sqrt{1-t^2}\sqrt{1-z^2}\,\chi^{-1}. \qquad (16.14)$$

With this, equation (16.13) implies that

$$\begin{aligned}\omega_t &= \frac{(1-z^2)}{\Psi^2}\left(\Phi_z - i(H\bar{H}_z - \bar{H}H_z)\right)\\ \omega_z &= \frac{(1-t^2)}{\Psi^2}\left(\Phi_t - i(H\bar{H}_t - \bar{H}H_t)\right).\end{aligned} \qquad (16.15)$$

These equations are integrable provided

$$\begin{aligned}\left(\frac{(1-t^2)}{\Psi^2}\left(\Phi_t - i(H\bar{H}_t - \bar{H}H_t)\right)\right)_{,t} \\ - \left(\frac{(1-z^2)}{\Psi^2}\left(\Phi_z - i(H\bar{H}_z - \bar{H}H_z)\right)\right)_{,z} = 0.\end{aligned} \qquad (16.16)$$

Also, with the substitutions (16.14) and (16.15), (16.11a) becomes

$$\begin{aligned}\left(\frac{(1-t^2)}{\Psi}\Psi_t\right)_{,t} - \left(\frac{(1-z^2)}{\Psi}\Psi_z\right)_{,z}& \\ = -\frac{(1-t^2)}{\Psi^2}\left(\Phi_t - i(H\bar{H}_t - \bar{H}H_t)\right)^2 &- \frac{2(1-t^2)}{\Psi^2}H_t\bar{H}_t \\ + \frac{(1-z^2)}{\Psi^2}\left(\Phi_z - i(H\bar{H}_z - \bar{H}H_z)\right)^2 &+ \frac{2(1-z^2)}{\Psi^2}H_z\bar{H}_z\end{aligned} \tag{16.17}$$

and Maxwell's equation (16.7) can be written in the form

$$\begin{aligned}\left(\frac{(1-t^2)}{\Psi}H_t\right)_{,t} - \left(\frac{(1-z^2)}{\Psi}H_z\right)_{,z} &= \frac{(1-t^2)}{\Psi^2}H_t(i\Phi_t + H\bar{H}_t - \bar{H}H_t) \\ &\quad - \frac{(1-z^2)}{\Psi^2}H_z(i\Phi_z + H\bar{H}_z - \bar{H}H_z).\end{aligned} \tag{16.18}$$

It is now convenient to generalize the definition (12.39) and to put

$$Z = \Psi + i\Phi + H\bar{H}. \tag{16.19}$$

With this, equation (16.18) can immediately be written in the form

$$\begin{aligned}(\mathcal{R}e\, Z - H\bar{H})\Big\{\big((1-t^2)H_t\big)_{,t} - \big((1-z^2)H_z\big)_{,z}\Big\}& \\ = (1-t^2)H_tZ_t - (1-z^2)H_zZ_z& \\ - 2\bar{H}\left\{(1-t^2)H_t^2 - (1-z^2)H_z^2\right\}&\end{aligned} \tag{16.20}$$

and, using (16.20), the main equations in the form (16.16) and (16.17) can be written as the single complex equation

$$\begin{aligned}(\mathcal{R}e\, Z - H\bar{H})\Big\{\big((1-t^2)Z_t\big)_{,t} - \big((1-z^2)Z_z\big)_{,z}\Big\}& \\ = (1-t^2)Z_t^2 - (1-z^2)Z_z^2& \\ - 2\bar{H}\left\{(1-t^2)H_tZ_t - (1-z^2)H_zZ_z\right\}.&\end{aligned} \tag{16.21}$$

These two equations are Ernst's equations, and are in fact identical to those for stationary axisymmetric Einstein–Maxwell fields. They involve potential functions from which the metric function χ can immediately be obtained using (16.14) and (16.19), and ω can be obtained by integrating (16.15). The electromagnetic field components can then be obtained using (16.8). When $H = 0$, this reduces to (11.18).

It may also be noted that equations (16.20) and (16.21) can be written in the coordinate-independent way

$$\begin{aligned}(\mathcal{R}e\, Z - H\bar{H})\nabla^2 Z &= (\nabla Z)^2 - 2\bar{H}(\nabla H).(\nabla Z)\\ (\mathcal{R}e\, Z - H\bar{H})\nabla^2 H &= (\nabla Z).(\nabla H) - 2\bar{H}(\nabla H)^2.\end{aligned} \tag{16.22}$$

These are the complex Ernst equations for an Einstein–Maxwell field which generalize the form (11.8).

An alternative form of these equations which generalizes (11.19) can be obtained by putting

$$Z = \frac{1+E}{1-E}, \qquad H = \frac{\eta}{1-E}, \tag{16.23}$$

or inversely

$$E = \frac{Z-1}{Z+1}, \qquad \eta = \frac{2H}{Z+1}. \tag{16.24}$$

With these definitions, (16.20) and (16.21) can be rewritten in the form

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})&\Big\{\big((1-t^2)E_t\big)_{,t} - \big((1-z^2)E_z\big)_{,z}\Big\}\\ &= -2\bar{E}\left\{(1-t^2)E_t^2 - (1-z^2)E_z^2\right\}\\ &\quad - 2\bar{\eta}\left\{(1-t^2)\eta_t E_t - (1-z^2)\eta_z E_z\right\}\end{aligned} \tag{16.25}$$

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})&\Big\{\big((1-t^2)\eta_t\big)_{,t} - \big((1-z^2)\eta_z\big)_{,z}\Big\}\\ &= -2\bar{\eta}\left\{(1-t^2)\eta_t^2 - (1-z^2)\eta_z^2\right\}\\ &\quad - 2\bar{E}\left\{(1-t^2)\eta_t E_t - (1-z^2)\eta_z E_z\right\}.\end{aligned} \tag{16.26}$$

These equations can also be written in the coordinate-independent way

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})\nabla^2 E &= -2\nabla E(\bar{E}\nabla E + \bar{\eta}\nabla\eta)\\ (1 - E\bar{E} - \eta\bar{\eta})\nabla^2 \eta &= -2\nabla\eta(\bar{E}\nabla E + \bar{\eta}\nabla\eta).\end{aligned} \tag{16.27}$$

For any solution of (16.20) and (16.21), or equivalently of (16.25) and (16.26), to be acceptable as a colliding plane wave solution, the appropriate boundary conditions must be satisfied. These may be taken in either of the forms (7.15) or (7.16). Alternatively, generalizing the form (11.26), the boundary conditions may be expressed in the form (Griffiths, 1990*a,b*)

$$\begin{aligned}\lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{Z_p\bar{Z}_p - 2(\bar{H}H_p\bar{Z}_p + H\bar{H}_pZ_p) + 2(Z+\bar{Z})H_p\bar{H}_p}{(Z+\bar{Z}-2H\bar{H})^2}\right] &= 2k_1\\ \lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{Z_q\bar{Z}_q - 2(\bar{H}H_q\bar{Z}_q + H\bar{H}_qZ_q) + 2(Z+\bar{Z})H_q\bar{H}_q}{(Z+\bar{Z}-2H\bar{H})^2}\right] &= 2k_2\end{aligned} \tag{16.28}$$

where, for convenience, we have put

$$\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial q} = \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \tag{16.29}$$

and k_1 and k_2 are given by (7.12), and are restricted to the range (7.13).

In terms of the alternative Ernst potential, these conditions can be expressed in the equivalent form

$$\begin{aligned} \lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(1-\eta\bar\eta)E_p\bar E_p + \bar\eta E\eta_p\bar E_p + \eta\bar E\bar\eta_p E_p + (1-E\bar E)\eta_p\bar\eta_p}{(1-E\bar E-\eta\bar\eta)^2} \right] &= 2k_1 \\ \lim_{\substack{t\to 0\\ z\to 0}} \left[\frac{(1-\eta\bar\eta)E_q\bar E_q - \bar\eta E\eta_q\bar E_q - \eta\bar E\bar\eta_q E_q + (1-E\bar E)\eta_q\bar\eta_q}{(1-E\bar E-\eta\bar\eta)^2} \right] &= 2k_2. \end{aligned} \tag{16.30}$$

Having found an acceptable solution for Z and H, or alternatively for E and η, it is finally necessary to determine the remaining metric function M. This can be achieved by integrating equations (7.9) in an appropriate form and using (7.8).

16.2 A simple class of solutions

It has been observed by Chandrasekhar and Xanthopoulos (1987*a*), that equation (16.21) is automatically satisfied when Z is a constant. It may also be noticed that, in this case, the remaining equation (16.20) does not contain the imaginary part of Z, and a constant imaginary part of Z would only yield a constant Φ which would have no effect on the metric through (16.15). It is therefore possible to consider Z as a real constant. In addition, replacing Z by $\alpha^2 Z$ and H by αH, where α is a real constant, leaves (16.20) unchanged. Thus, once it is assumed that Z is a constant, no loss of generality is entailed by the assumption that

$$Z = 1. \tag{16.31}$$

In this case (16.20) becomes

$$\begin{aligned} (1-H\bar H)\Big(\big((1-t^2)H_t\big)_{,t} - \big((1-z^2)H_z\big)_{,z}\Big) \\ = -2\bar H\left((1-t^2)H_t^2 - (1-z^2)H_z^2\right) \end{aligned} \tag{16.32}$$

which is identical in form to (11.19).

Thus the assumption (16.31) requires that H satisfies the alternative form of the Ernst equation for a vacuum. This enables numerous solutions

to be easily generated. All that is required is a solution E_o of the vacuum Ernst equation (11.19). Then, a new electromagnetic solution is obtained simply by putting

$$Z = 1, \qquad H = E_o. \tag{16.33}$$

Expressing this in terms of the alternative Ernst functions, if E_o is a solution of the vacuum Ernst equation (11.14), then a new electromagnetic solution of (16.25,26) is obtained simply by putting

$$E = 0, \qquad \eta = E_o. \tag{16.34}$$

Moreover, by comparing (16.28) with (11.27) it is clear that, if the boundary conditions for colliding waves are satisfied for the vacuum solution E_o, then the appropriate boundary conditions are automatically satisfied for the new solution (16.33).

With (16.33), it follows that

$$\Psi = 1 - E_o\bar{E}_o, \qquad \Phi = 0, \qquad \chi = \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{1-E_o\bar{E}_o} \tag{16.35}$$

and ω can be found by integrating (16.15), which may be rewritten as

$$\begin{aligned} \omega_t &= -i\frac{(1-z^2)}{(1-E_o\bar{E}_o)^2}(E_o\bar{E}_{oz} - \bar{E}_oE_{oz}) \\ \omega_z &= -i\frac{(1-t^2)}{(1-E_o\bar{E}_o)^2}(E_o\bar{E}_{ot} - \bar{E}_oE_{ot}). \end{aligned} \tag{16.36}$$

The remaining metric function M may finally be determined by integrating an appropriate transformation of equations (7.9). However, the resulting equations will not be a generalization of (11.21), since in that equation the function E was related to χ and ω rather than Ψ and Φ. In this case (7.9) may be written in the form

$$\begin{aligned} S_f &= -\frac{1}{2(f+g)} - \frac{E_o\bar{E}_{of} + \bar{E}_oE_{of}}{(1-E_o\bar{E}_o)} - \frac{2(f+g)E_{of}\bar{E}_{of}}{(1-E_o\bar{E}_o)^2} \\ S_g &= -\frac{1}{2(f+g)} - \frac{E_o\bar{E}_{og} + \bar{E}_oE_{og}}{(1-E_o\bar{E}_o)} - \frac{2(f+g)E_{og}\bar{E}_{og}}{(1-E_o\bar{E}_o)^2}. \end{aligned} \tag{16.37}$$

It may be recalled that E_o is any solution of the vacuum Ernst equation (11.14). Thus, for any initial vacuum solution expressed in terms of the potentials Ψ_o and Φ_o, an electromagnetic solution can be obtained by the above method using

$$E_o = \frac{\Psi_o + i\Phi_o - 1}{\Psi_o + i\Phi_o + 1}, \qquad \Psi = \frac{4\Psi_o}{(\Psi_o+1)^2 + \Phi_o^2}, \qquad \Phi = 0. \tag{16.38}$$

In addition, (16.37) may now be written in the form

$$\begin{aligned} S_f &= -\frac{1}{2(f+g)} - \frac{\Psi_f}{\Psi} - \frac{(f+g)({\Psi_\mathrm{o}}_f^{\,2}+{\Phi_\mathrm{o}}_f^{\,2})}{2\Psi_\mathrm{o}^2} \\ S_g &= -\frac{1}{2(f+g)} - \frac{\Psi_g}{\Psi} - \frac{(f+g)({\Psi_\mathrm{o}}_g^{\,2}+{\Phi_\mathrm{o}}_g^{\,2})}{2\Psi_\mathrm{o}^2}. \end{aligned} \tag{16.39}$$

By comparing this with (12.41) it may be seen that, if the initial vacuum solution contains the metric function M_o, then equations (16.39) can be integrated to give

$$e^{-M} = \frac{\Psi_\mathrm{o}}{\Psi} e^{-M_\mathrm{o}} = \tfrac{1}{4}\big((\Psi_\mathrm{o}+1)^2 + \Phi_\mathrm{o}^2\big) e^{-M_\mathrm{o}}. \tag{16.40}$$

It also follows from this that

$$U = U_\mathrm{o} \tag{16.41}$$

only if $\big((\Psi_\mathrm{o}+1)^2+\Phi_\mathrm{o}^2\big)$ is continuous across the boundaries of region IV.

16.3 The Bell–Szekeres solution

In order to describe the Bell–Szekeres solution in the present notation, it may immediately be observed by substituting (15.6) into (16.4), and using (11.2) that

$$H = \sin(au-bv), \qquad \chi = \frac{\cos(au+bv)}{\cos(au-bv)}, \qquad \omega = 0. \tag{16.42}$$

From (16.14), it follows that $\Psi = \cos^2(au-bv)$, and clearly $\Phi = 0$. The Ernst potentials for the Bell–Szekeres solution are therefore given by

$$Z = 1, \qquad H = z. \tag{16.43}$$

The alternative Ernst potentials are

$$E = 0, \qquad \eta = z. \tag{16.44}$$

This clearly satisfies the boundary conditions (16.28) with $k_1 = k_2 = \frac{1}{2}$ as required.

It has frequently been noted that V can always be replaced by $-V$. Such a transformation, in this case, would yield a solution in which

$$H = \sin(au+bv), \qquad \chi = \frac{\cos(au-bv)}{\cos(au+bv)}, \qquad \omega = 0 \tag{16.45}$$

and therefore

$$H = t, \qquad Z = 1, \qquad \Phi_0^\circ = -b, \qquad \Phi_2^\circ = a. \tag{16.46}$$

The potentials (16.43) and (16.46) seem to indicate that the Bell–Szekeres solution may be included in a solution obtained by the method of the previous section, in which $Z = 1$ and H is taken to be the Ernst function (13.4) of the Nutku–Halil solution: namely

$$H = pt + iqz \tag{16.47}$$

where $p^2 + q^2 = 1$. It may be noted that, when $p = 0$, a change from $H = z$ to $H = iz$ in (16.43) would yield $\Phi_0^\circ = ib$ and $\Phi_2^\circ = ia$ which are related to the components of (15.6) by a simple duality rotation.

The potential (16.47), with $Z = 1$, implies that

$$\Psi = 1 - p^2t^2 - q^2z^2, \qquad \Phi = 0, \qquad \chi = \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{1 - p^2t^2 - q^2z^2} \tag{16.48}$$

and equations (16.36) can be integrated to give

$$\omega = \frac{p}{q}\frac{(1-t^2)}{(1 - p^2t^2 - q^2z^2)} + \text{const.} \tag{16.49}$$

This solution has been considered in detail by Chandrasekhar and Xanthopoulos (1987*a*), who have taken the constant of integration in (16.49) to be zero so that $\omega = 0$ when $t = 1$. They have then shown that the resulting solution is a simple transformation of the conformally flat Bell–Szekeres solution.

Alternatively, taking the constant of integration in (16.49) to be $-p/q$ gives the rotation of the Bell–Szekeres solution described in Section 15.4, with the parameters related to that of (15.18) by

$$p = \cos\frac{\alpha}{2}, \qquad q = -\sin\frac{\alpha}{2}. \tag{16.50}$$

In either case, it may be concluded that the solution arising from (16.47) is the Bell–Szekeres solution, but with the polarization of the step electromagnetic waves being non-aligned.

It may also be observed that the general solution above can be obtained by the methods of the previous section with the initial vacuum solution given by

$$E_\mathrm{o} = \frac{\Psi_\mathrm{o} + i\Phi_\mathrm{o} - 1}{\Psi_\mathrm{o} + i\Phi_\mathrm{o} + 1} = pt + iqz \tag{16.51}$$

which is equivalent to the Ernst potential (13.10) that gives rise to the vacuum Chandrasekhar–Xanthopoulos solution. On substituting these expressions, (16.40) takes the form

$$e^{-M} = \frac{1}{(1-pt)^2 + q^2z^2} e^{-M_\mathrm{o}} \tag{16.52}$$

which, from (13.20), becomes

$$e^{-M} = \frac{1}{\sqrt{1-u^2}\sqrt{1-v^2}}. \tag{16.53}$$

It follows from this that $U = U_\mathrm{o} = -\log(1-u^2-v^2)$. However, it is clearly appropriate to make the transformation

$$u \to \sin au, \qquad v \to \sin bv \tag{16.54}$$

which, with a scale factor, gives the familiar form of the Bell–Szekeres solution.

17

COLLIDING ELECTROMAGNETIC WAVES: EXACT SOLUTIONS

It is the purpose of this chapter and the following one to review the exact solutions which describe colliding plane electromagnetic waves or combinations of gravitational and electromagnetic waves that are presently known. Since these space-times have two Killing vectors and an Ernst method can be used, there are a whole range of generating techniques that can be applied to obtain new classes of solutions. It is therefore not possible to review exact solutions without also reviewing the generating techniques that have been applied to obtain them. The techniques described may clearly be used to obtain further solutions. This chapter reviews classes of non-diagonal solutions. The particular case of diagonal solutions is considered in Chapter 18.

17.1 A technique of Chandrasekhar and Xanthopoulos

Having developed the approach described in the previous chapter, Chandrasekhar and Xanthopoulos (1985*a*) then used it to obtain an electromagnetic generalization of the Nutku–Halil solution.

If the Khan–Penrose and Nutku–Halil solutions are regarded as the analogues of the Schwarzschild and Kerr solutions respectively, then it is reasonable to suppose that the Kerr–Newman solution may also have a colliding wave analogue. Accordingly, it is appropriate[1] to consider the special class of solutions for which

$$H = Q(Z+1) \tag{17.1}$$

where Q is a constant that can always be chosen to be real. In this case, (16.20) and (16.21) both reduce to the same equation:

$$\begin{aligned}\tfrac{1}{2}\left((1-2Q^2)(Z+\bar{Z})-2Q^2(Z\bar{Z}+1)\right)\Big(\big((1-t^2)Z_t\big)_{,t}-\big((1-z^2)Z_z\big)_{,z}\Big)\\ =\left(1-2Q^2(\bar{Z}+1)\right)\left((1-t^2)Z_t^2-(1-z^2)Z_z^2\right).\end{aligned} \tag{17.2}$$

[1] See Chandrasekhar (1983) section 110.

It is also convenient to consider the restriction (17.1) in terms of the associated Ernst potentials E and η given by (16.23,24). It can immediately be seen that, in this case,

$$\eta = 2Q \tag{17.3}$$

so that equation (16.25) is automatically satisfied, and (16.25) becomes

$$\begin{aligned}(1-4Q^2-E\bar{E})\Big(\big((1-t^2)E_t\big)_{,t}-\big((1-z^2)E_z\big)_{,z}\Big)&\\ =-2\bar{E}\left((1-t^2)E_t^2-(1-z^2)E_z^2\right)&\end{aligned} \tag{17.4}$$

which is virtually identical to (11.19).

In this case, however, it is important to notice that the functions Z and E do not contain the metric functions as in Chapter 11. Rather, through (16.19), they contain the potentials described in the previous chapter. These are now given in terms of E by

$$\Psi=\frac{(1-4Q^2-E\bar{E})}{(1-E)(1-\bar{E})},\qquad \Phi=-i\frac{(E-\bar{E})}{(1-E)(1-\bar{E})},\qquad H=\frac{2Q}{1-E}. \tag{17.5}$$

Chandrasekhar and Xanthopoulos (1985*a*) have made use of the important result that, if $E_{\rm o}$ is a solution of the vacuum Ernst equation (11.14), then

$$E = aE_{\rm o} \tag{17.6}$$

is a solution of the equation (17.4) for an Einstein–Maxwell field, where

$$a=\sqrt{1-4Q^2}. \tag{17.7}$$

In view of the possible transformation (12.9), there is no loss of generality in assuming that a is real and that $0 \leq Q \leq \frac{1}{2}$. This technique can therefore be characterized by the Ernst potentials

$$E = aE_{\rm o}, \qquad \eta=\sqrt{1-a^2}. \tag{17.8}$$

This result can also be stated in the equivalent form that, if $Z_{\rm o}$ is a solution of the vacuum Ernst equation (11.8), then a new solution of the Ernst equations (16.20) and (16.21) for an Einstein–Maxwell field is given by

$$Z=\frac{(1-a)+(1+a)Z_{\rm o}}{(1+a)+(1-a)Z_{\rm o}},\qquad H=\frac{\sqrt{1-a^2}\;(Z_{\rm o}+1)}{(1+a)+(1-a)Z_{\rm o}}. \tag{17.9}$$

It follows immediately from (17.9) that the potentials for the new solution are given in terms of the vacuum potentials $Z_o = \Psi_o + i\Phi_o$ by

$$\begin{aligned} \Psi &= \frac{2a^2}{\Omega^2}(Z_o + \bar{Z}_o) \quad = \frac{4a^2}{\Omega^2}\Psi_o \\ \Phi &= -\frac{2ia}{\Omega^2}(Z_o - \bar{Z}_o) = \frac{4a}{\Omega^2}\Phi_o \end{aligned} \tag{17.10}$$

where

$$\begin{aligned} \Omega^2 &= |(1+a) + (1-a)Z_o|^2 \\ &= (1+a)^2 + 2(1-a^2)\Psi_o + (1-a)^2(\Psi_o^2 + \Phi_o^2). \end{aligned} \tag{17.11}$$

Clearly, it is possible to use this method to obtain new solutions of the Einstein–Maxwell equations starting with any vacuum solution describing colliding gravitational waves. The new solutions contain an arbitrary parameter a. Moreover, the new solutions must reduce to the original vacuum solutions in the limit as $a \to 1$.

By substituting (17.8) into (16.28), it may be observed that the boundary conditions for colliding gravitational and electromagnetic waves of this type are automatically satisfied if they were satisfied for the original vacuum solution given by E_o or Z_o, and with the same values for k_1 and k_2. This implies that the expressions for $f(u)$ and $g(v)$ are unchanged, and thus

$$U = U_o. \tag{17.12}$$

Using the approach described above with (17.1) and (17.6), the metric function χ can immediately be obtained from (16.14). It is then necessary to integrate the equations (16.15) to obtain ω. Finally it is necessary to integrate (7.9) to determine the remaining metric function M.

It may be noticed, however, that the transformation (17.9) is contained in the 'Ehlers–Harrison' transformation for a particular choice of parameters (Harrison, 1968). In this case, it is possible to find a general transformation for M, so that an integration of (7.9) is not essential. But there does not appear to be any way of avoiding the integration of (16.15) to obtain ω.

17.2 Two particular examples

In this section two results are presented that have been obtained using the above technique. The first of these is a generalization of the Nutku–Halil solution and the second a generalization of the vacuum Chandrasekhar–Xanthopoulos solution.

Chandrasekhar and Xanthopoulos (1985*a*) have obtained a new colliding wave solution using the above method and taking the initial vacuum solution to be the Nutku–Halil solution described in Section 13.1. In this case, we have $Z_o = \Psi_o + i\Phi_o$ where

$$\begin{aligned} \Psi_o &= \sqrt{1-t^2}\sqrt{1-z^2}\left(\frac{(1-pt)^2+q^2z^2}{1-p^2t^2-q^2z^2}\right) \\ \Phi_o &= \frac{2q}{p}\frac{(1-z^2)(1-pt)}{(1-p^2t^2-q^2z^2)}. \end{aligned} \tag{17.13}$$

These expressions may immediately be deduced from equations (13.11) and (13.17) after noting that the vacuum Chandrasekhar–Xanthopoulos and the Nutku–Halil solutions are related simply by interchanging the expressions for Ψ and Φ with those for χ and ω. In this case, it may also be noticed that a constant term in Φ_o has been altered.

The metric functions for this solution can now be obtained from (16.14) and by integrating equations (16.15) and (7.9). The integration process is described in great detail by Chandrasekhar and Xanthopoulos (1985*a*). The resulting expressions, which are the metric functions of the line element (11.4), may be written in the form

$$\begin{aligned} e^{-U} &= \sqrt{1-t^2}\sqrt{1-z^2} = 1-u^2-v^2 \\ \chi &= \frac{\Omega^2}{4a^2}\frac{(1-p^2t^2-q^2z^2)}{((1-pt)^2+q^2z^2)} \\ \omega &= \frac{qz\left(4ap^2+(1-a)^2\left[1+q^2z^2+(3-z^2)(1-pt)^2\right]\right)}{2a^2p^2\left((1-pt)^2+q^2z^2\right)} \\ e^{-M} &= \frac{\Omega^2}{4a^2}\frac{(1-p^2t^2-q^2z^2)}{(1-t^2)^{1/4}(1-z^2)^{1/4}} \end{aligned} \tag{17.14}$$

where Ω is given in terms of the expressions (17.13) by (17.11).

The structure of this solution has been described in detail by Chandrasekhar and Xanthopoulos (1985*a*), with particular emphasis on the colinear case when $q = 0$ and $p = 1$. As usual, there is a strong curvature singularity in region IV on the hypersurface $u^2+v^2 = 1$, and there are fold singularities in regions II and III when $u = 1$ and $v = 1$. The approaching waves are a combination of step gravitational and electromagnetic components, and the wave fronts also include impulsive gravitational waves.

A second example of the application of the technique described in Section 17.1 is contained in part I of another substantial paper of Chandrasekhar and Xanthopoulos (1987*b*). This is obtained by taking the

initial solution to be the vacuum Chandrasekhar–Xanthopoulos (1986*b*) solution described in Section 13.3.

In this second example, the expression for Z_o is taken to be (13.10) so that

$$\Psi_o = \frac{1-p^2t^2-q^2z^2}{(1-pt)^2+q^2z^2}, \qquad \Phi_o = \frac{2qz}{(1-pt)^2+q^2z^2}. \tag{17.15}$$

Equations (16.15) may now be solved as in the previous example, and the resulting solution may be expressed in terms of the metric functions of the line element (11.4) as

$$\begin{aligned} e^{-U} &= \sqrt{1-t^2}\sqrt{1-z^2} = 1-u^2-v^2 \\ \chi &= \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{a^2}\left(\frac{(1-apt)^2+a^2q^2z^2}{1-p^2t^2-q^2z^2}\right) \\ \omega &= -\frac{q(1-z^2)}{a^2p}\left(\frac{1+a^2-2apt}{1-p^2t^2-q^2z^2}\right)+\text{const} \\ e^{-M} &= \frac{(1-apt)^2+a^2q^2z^2}{a^2\sqrt{1-u^2}\sqrt{1-v^2}}. \end{aligned} \tag{17.16}$$

Chandrasekhar and Xanthopoulos (1987*b*) have analysed this solution and have shown that it is of algebraic type D. They have further shown that it is in fact the Kerr–Newman space-time interior to its ergo-sphere. In this sense, it can be considered to be the natural generalization of the vacuum Chandrasekhar–Xanthopoulos (1986*b*) solution, which is a Kerr space-time interior to the ergo-sphere.

The singularity structure of this solution is the same as that of the vacuum Chandrasekhar–Xanthopoulos solution as described in Section 13.3. The space-like surface on which $u^2+v^2=1$ is an unstable quasiregular singularity rather than a curvature singularity.

In both of the solutions presented in this section there are impulsive gravitational wave components along the wavefronts $u=0$ and $v=0$. Hence these solutions do not satisfy the conditions for Tipler's theorem 8.1.

The global structure of this solution has been further considered by Hayward (1989*a*) as an extension of his analysis of the degenerate Ferrari–Ibañez solution.

17.3 Another type D solution

It has already been pointed out in Section 16.3 that the Bell–Szekeres solution can be obtained from the vacuum Chandrasekhar–Xanthopoulos

solution using the simple generation technique described in Section 16.2. Chandrasekhar and Xanthopoulos (1987*a*) have subsequently obtained a generalization of the Bell–Szekeres solution by applying an Ehlers transformation to the Ernst potential that is used to generate the Bell–Szekeres solution. The resulting space-time describing the interaction region of the new solution is of algebraic type D.

It may first be recalled that, if Z_{oo} is a solution of the vacuum Ernst equation (11.15), then another solution is given by

$$Z_o = \frac{Z_{oo}}{1 + 2i\beta Z_{oo}} \tag{17.17}$$

where the parameter of this Ehlers transformation has been altered from that of (12.13) by putting $c = -2\beta$. In terms of the associated Ernst function, the transformation (12.14) may be restated in the form that, if E_{oo} is a solution of equation (11.14), then a new vacuum solution can be obtained using the new potential

$$E_o = \frac{E_{oo} - i\beta(1 + E_{oo})}{1 + i\beta(1 + E_{oo})}. \tag{17.18}$$

Now, using the simple generation technique described in Section 16.2, an electromagnetic solution of the equations (16.20) and (16.21) is given by

$$Z = 1 \qquad \text{and} \qquad H = E_o \tag{17.19}$$

where E_o is a solution of the vacuum Ernst equation (11.14). The generalized non-diagonal Bell–Szekeres solution is expressed in this way in Section 16.3 with $E_o = pt + iqz$. A further generalization of the Bell–Szekeres solution may now be obtained by applying the Ehlers transformation (17.18) to this expression for E_o. Accordingly, we now consider an electromagnetic solution given by the potentials

$$Z = 1, \qquad H = \frac{E_{oo} - i\beta(1 + E_{oo})}{1 + i\beta(1 + E_{oo})} \tag{17.20}$$

where

$$E_{oo} = pt + iqz, \qquad \text{and} \qquad p^2 + q^2 = 1. \tag{17.21}$$

From these expressions it follows that

$$\Psi = \frac{1 - p^2t^2 - q^2z^2}{(1 - \beta qz)^2 + \beta^2(1 + pt)^2}, \qquad \Phi = 0. \tag{17.22}$$

The expression for χ is now determined by (16.14), and the remaining metric functions for this solution can be obtained by integrating equations (16.15) and (7.9). The details of the integration process are described by Chandrasekhar and Xanthopoulos (1987*a*), and the resulting expressions may be written in the form

$$\begin{aligned} e^{-U} &= \sqrt{1-t^2}\sqrt{1-z^2} = 1-u^2-v^2 \\ \chi &= \frac{\sqrt{1-t^2}\sqrt{1-z^2}\,\big((1-\beta qz)^2+\beta^2(1+pt)^2\big)}{(1-p^2t^2-q^2z^2)} \\ \omega &= \frac{p(1-t^2)\big(1-2\beta qz+2\beta^2(1+p)\big)+2\beta^2q^2(1-t)(1-z^2)}{q(1-p^2t^2-q^2z^2)} \\ e^{-M} &= (1-\beta qz)^2+\beta^2(1+pt)^2 \end{aligned} \tag{17.23}$$

where the arbitrary constant in the integral for ω has been chosen such that $\omega=0$ when $t=1$.

It may be noticed that this solution reduces to the conformally flat Bell–Szekeres solution in the limit as $\beta\to 0$. It may also be observed that it does not contain the special case of a diagonal solution in which $\omega=0$, except in this Bell–Szekeres limit.

The properties of this solution have been described in detail by Chandrasekhar and Xanthopoulos (1987*a*). They have evaluated the components of the curvature tensor and shown that the space-time in the interaction region is of algebraic type D, with the repeated principal null directions of the Weyl tensor being aligned with the principal null directions of the electromagnetic field. The approaching waves in regions II and III are a combination of gravitational and electromagnetic waves, and the usual fold singularities in these regions still occur.

One particularly significant feature of this solution is that the surface in region IV on which $t=1$ does not correspond to a curvature singularity. This surface, on which the two waves mutually focus each other, is a quasiregular singularity on which the components of the curvature tensor remain bounded. In this case, there are impulsive gravitational wave components along the wavefronts $u=0$ and $v=0$, and hence the solution does not satisfy the conditions for Tipler's theorem 8.1. In this feature it is similar to the Bell-Szekeres solution, the degenerate Ferrari–Ibañez solution of Section 10.5 and the vacuum Chandrasekhar–Xanthopoulos solution described in Section 13.3.

The solution given in this section has also been independently obtained and described by Halilsoy (1988*a*). Halilsoy has also used the simplification (17.19) as described in Section 16.2, but with the initial solution E_o being that obtained from a colinear solution using the transformation (12.44) and (12.45) with $a=1$. However, as is clear from

(12.46), this can also be considered as an Ehlers transformation (12.13) or (17.17) following a rotation of the type (12.10). It can be seen that this is precisely the sequence of transformations used by Chandrasekhar and Xanthopoulos (1987*a*) as described above.

17.4 A technique of Halilsoy

A simple technique was described in Section 16.2 in which a solution $E_{\rm o}$ of the vacuum Ernst equation (11.14) was used to generate a colliding electromagnetic wave solution of the equations (16.21) and (16.20) given by $Z=1$, $H=E_{\rm o}$. This technique was included in that of the previous section. In terms of the alternative Ernst potentials, the new solution satisfying (16.25) and (16.26) is given by $E=0$, $\eta=E_{\rm o}$.

As pointed out by Halilsoy (1989*a*,*b*), this generation technique can easily be generalized. It can immediately be seen that a new colliding electromagnetic wave solution of the equations (16.25) and (16.26) can be given in terms of the alternative Ernst potentials by

$$E=aE_{\rm o}, \qquad \eta=\sqrt{1-a^2}\,E_{\rm o} \tag{17.24}$$

where a is an arbitrary parameter.

It is also sometimes convenient to express this transformation in terms of an alternative parameter α such that

$$a=\cos 2\alpha. \tag{17.25}$$

The new solution may then be expressed in the form

$$E=\cos 2\alpha\, E_{\rm o}, \qquad \eta=\sin 2\alpha\, E_{\rm o}. \tag{17.26}$$

In terms of the original Ernst potentials, the transformation (17.24) or (17.26) can also be stated in the equivalent form that, if $Z_{\rm o}$ is a solution of the vacuum Ernst equation (11.8), then a new electromagnetic solution satisfying (16.21) and (16.20) is given by

$$\begin{aligned} Z&=\frac{(1-a)+(1+a)Z_{\rm o}}{(1+a)+(1-a)Z_{\rm o}}=\frac{\sin^2\alpha+\cos^2\alpha\, Z_{\rm o}}{\cos^2\alpha+\sin^2\alpha\, Z_{\rm o}}\\ H&=\frac{\sqrt{1-a^2}\,(Z_{\rm o}-1)}{(1+a)+(1-a)Z_{\rm o}}=\frac{\sin\alpha\cos\alpha\,(Z_{\rm o}-1)}{\cos^2\alpha+\sin^2\alpha\, Z_{\rm o}}. \end{aligned} \tag{17.27}$$

It is of interest to note the similarity between the initial forms of these expressions and those of (17.9).

Proceeding in a similar fashion as in (17.10), it can be seen that the potentials for the new solution are given in terms of the vacuum potentials $Z_o = \Psi_o + i\Phi_o$ by

$$\Psi = \frac{4}{\Omega^2}\Psi_o \qquad \Phi = \frac{4a}{\Omega^2}\Phi_o \tag{17.28}$$

where, as in (17.9),

$$\begin{aligned}\Omega^2 &= |(1+a) + (1-a)Z_o|^2 \\ &= 4|\cos^2\alpha + \sin^2\alpha\, Z_o|^2.\end{aligned} \tag{17.29}$$

The expressions (17.27) may be substituted directly into (16.28), which then reduces exactly to (11.26). It follows that, if the initial solution Z_o satisfies the boundary conditions for colliding plane gravitational waves, then the new solution (17.25) also satisfies the required boundary conditions for the same metric function U given by

$$U = U_o. \tag{17.30}$$

It can be seen that the vacuum solution can be recovered when $a = 1$ or $\alpha = 0$, and that the original technique of Section 16.2 which is given by (16.33) or (16.34) is obtained when $a = 0$ or $\alpha = \pi/4$.

The transformation (17.24) was first used by Halilsoy (1989*a*) to obtain a new class of solutions with the familiar form for the vacuum solution given by

$$E_o = pt + iqz \tag{17.31}$$

where, as in (13.4), $p^2+q^2 = 1$. The resulting solutions have been analysed in detail. In the limit as $a \to 0$, this solution must reduce to that of Chandrasekhar and Xanthopoulos (1986*b*) as described in Section 13.3, which is the Kerr solution in the interaction region.

It is also of interest to note that it is a very simple matter to generalize the transformation (17.24), or (17.26), that has been used in this section. It can easily be seen that, if E_o is a solution of the vacuum equation (11.14), then a family of solutions of the electrovac equations (16.25) and (16.26) is given by

$$E = aE_o + b, \qquad \eta = cE_o + d \tag{17.32}$$

where a, b, c and d are arbitrary complex constants that are only required to satisfy the two conditions

$$a\bar{b} + c\bar{d} = 0, \qquad a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} = 1. \tag{17.33}$$

It can immediately be seen that this generation technique is not only a generalization of (17.24) and hence also of (16.33), but is also a generalization of the transformation (17.8) that was described in Section 17.1 and applied in Section 17.2. The generation technique given by (17.32) may also be expressed in the equivalent form

$$Z = \frac{1+b+aE_o}{1-b-aE_o}, \qquad H = \frac{d+cE_o}{1-b-aE_o}. \tag{17.34}$$

By substituting (17.32) into (16.30), it can be seen that the boundary conditions for colliding plane waves are automatically satisfied for the new solution if they are satisfied for the initial solution E_o. It follows that, using the transformation (17.32), a large class of solutions describing colliding combined electromagnetic and gravitational waves can be obtained from any colliding gravitational plane wave solution.

17.5 Other solutions

It has been pointed out by Neugebauer and Kramer (1969) that, for any space-time having a non-null Killing vector, there are a number of invariance transformations for Einstein–Maxwell fields on the assumption that the electromagnetic field shares the space-time symmetry. These transformations can be considered to arise from the existence of scalar potentials and the symmetries of the associated potential space. This is well reviewed by Kramer *et al.* (1980). It is appropriate at the beginning of this section simply to summarize a number of results which follow from this, and which apply specifically to the colliding plane wave problem in which there is an additional space-like Killing vector.

In terms of the Ernst potentials introduced above, the invariance transformations may be stated in the form that if (Z_o, H_o) is a solution of the Ernst equations (16.22), then other solutions (Z, H) are given by

$$Z = \alpha\bar{\alpha}Z_o, \qquad H = \alpha H_o, \tag{17.35a}$$

$$Z = Z_o + ib, \qquad H = H_o, \tag{17.35b}$$

$$Z = \frac{Z_o}{1+icZ_o}, \qquad H = \frac{H_o}{1+icZ_o}, \tag{17.35c}$$

$$Z = Z_o + 2\bar{\beta}H_o + \beta\bar{\beta}, \qquad H = H_o + \beta, \tag{17.35d}$$

$$Z = \frac{Z_o}{1+2\bar{\gamma}H_o+\gamma\bar{\gamma}Z_o}, \qquad H = \frac{H_o+\gamma Z_o}{1+2\bar{\gamma}H_o+\gamma\bar{\gamma}Z_o} \tag{17.35e}$$

where α, β, and γ are complex constants and b and c are real constants. Together, these constants form the eight real parameters of the isometry

group of the potential space. Clearly, these transformations can be used in any combination to generate new solutions of the Ernst equation.

It may immediately be observed that, in the absence of the electromagnetic field, (17.35*b*) is the transformation (12.17) and (17.35*c*) is the Ehlers transformation (12.13). Also, if the initial solution is purely gravitational so that $H_o = 0$, then any combination of these transformations which includes (17.35*e*) will generate electromagnetic solutions from the initial vacuum solutions. Applied in this sense with $H_o = 0$, (17.35*e*) is referred to as a Harrison transformation (Harrison, 1968).

It can also be shown that, if the initial solution (Z_o, H_o) satisfies the boundary conditions (16.28) for colliding plane waves, then this condition is also satisfied by a new solution (Z, H) obtained by any of the invariance transformations (17.35) for the same values of the parameters k_i. Thus, for any initial solution which is a colliding plane wave solution, further colliding plane wave solutions can be generated by these transformations. This result is very remarkable since, in the context of stationary axisymmetric space-times, these transformations do not always preserve asymptotic flatness and so are not of great practical importance in that situation.

It was initially pointed out by Kinnersley (1973)[2] that the group of symmetry transformations of the Einstein–Maxwell equations with a non-null Killing vector is the group $SU(2,1)$. This is the group of the transformations (17.35). It has been further described in terms of colliding plane wave solutions by Li and Ernst (1989).

It can now be seen that the approaches of Chandrasekhar and Xanthopoulos (1985*a*) described in Section 17.1 and of Halilsoy (1989*a*) as described in Section 17.4 are both equivalent to 'Ehlers–Harrison' transformations for particular choices of parameters.

Clearly a large class of further colliding plane wave solutions can be obtained using the above transformations. A four-parameter generalization of the Nutku–Halil solution has already been obtained by García-Díaz (1988) using this method. García-Díaz (1989) has also used a Harrison transformation to obtain a four-parameter generalization of the Ferrari–Ibañez solution. Further solutions have been generated in this way by Li and Ernst (1989) generalizing the solutions of Ernst, García-Díaz and Hauser (1987*a,b*, 1988), Ferrari, Ibañez and Bruni (1987*a,b*) and the Nutku–Halil solutuion.

There is also a large number of more general solution-generating techniques that can be applied to this situation. These are effectively generalizations of those mentioned briefly in Section 12.6, which were initially

[2] See also Kramer *et al.* (1980) section 30.3.

developed in the context of stationary axisymmetric space-times. For example, Cosgrove (1981) has found a Bäcklund transformation, expressed in terms of the Hauser–Ernst formalism, which applies to electrovac space-times with two Killing vectors.

It is finally appropriate here to refer to other non-diagonal solutions which describe colliding plane electromagnetic waves.

A five-parameter family of type D electrovac solutions has been given by Papacostas and Xanthopoulos (1989). In these solutions, the hypersurface $f+g=0$ in region IV is a quasi–regular singularity. The solution is in fact a special case of a family of solutions given by Debever (1971), Plebański (1975) and Plebański and Demiański (1976).[3] There is also a vacuum limit of this solution which describes a collision of gravitational waves.

[3] See also Kramer *et al.* (1980) section 19.1.

18

COLLIDING ELECTROMAGNETIC WAVES: DIAGONAL SOLUTIONS

This chapter is a continuation of the review of all presently known exact solutions which describe the collision of plane electromagnetic waves, or a combination of both gravitational and electromagnetic waves. Attention is concentrated here only on diagonal solutions. These solutions may be considered as a generalization of the solutions representing the collision of gravitational waves with colinear aligned polarization that have been described in Chapter 10. It may be mentioned that the Bell–Szekeres solution that has previously been described in Chapter 15 also belongs to the class that is now being surveyed.

18.1 The generation technique of Panov

Panov (1978, 1979*a*) has obtained a class of exact solutions for colliding gravitational-electromagnetic waves using a generating technique due to Enss (1967). His method starts with a vacuum solution with aligned polarization, and can conveniently be described as follows.

Taking the metric in the form of the Szekeres line-element (6.20), if a solution of (6.22) describing a collision of plane colinear gravitational waves is given by U_o, V_o, and M_o with $W_\mathrm{o}=0$, then a new solution with non-zero electromagnetic components is given by

$$\begin{aligned} U &= U_\mathrm{o}, \qquad\qquad W = W_\mathrm{o} = 0 \\ e^V &= e^{V_\mathrm{o}}\left(\cos^2\alpha + \sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)^2 \\ e^{-M} &= e^{-M_\mathrm{o}}\left(\cos^2\alpha + \sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)^2 \end{aligned} \tag{18.1}$$

where α is an arbitrary constant. The Maxwell components are given by

$$\begin{aligned} \Phi_0^\circ &= -i\frac{\sin\alpha\cos\alpha e^{-(U_\mathrm{o}+V_\mathrm{o})/2}}{\left(\cos^2\alpha + \sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)}(U_{\mathrm{o}v}+V_{\mathrm{o}v}) \\ \Phi_2^\circ &= -i\frac{\sin\alpha\cos\alpha e^{-(U_\mathrm{o}+V_\mathrm{o})/2}}{\left(\cos^2\alpha + \sin^2\alpha\, e^{-(U_\mathrm{o}+V_\mathrm{o})}\right)}(U_{\mathrm{o}u}+V_{\mathrm{o}u}). \end{aligned} \tag{18.2}$$

It may immediately be seen from these equations that the boundary conditions for colliding plane waves are automatically satisfied for the new solution if they were satisfied for the original vacuum solution. The new solution therefore describes the collision of combined gravitational and electromagnetic waves with aligned polarization. The electromagnetic components vanish and the solution reverts to the original when $\alpha = 0$.

In terms of the notation developed in Chapter 16, the initial solution may be described either by the function χ_o with $\omega_o = 0$, or by Ψ_o with $\Phi_o = 0$. The new solution is then given by

$$\chi = \chi_o \left(\cos^2\alpha + \sin^2\alpha\, e^{-U_o}\chi_o\right)^{-2}, \qquad \omega = 0, \tag{18.3}$$

and the potential H can be obtained by integrating the equations

$$\begin{aligned} H_v &= i\sin\alpha\cos\alpha\,(U_{ov} + V_{ov})\, e^{-U_o} \\ H_u &= -i\sin\alpha\cos\alpha\,(U_{ou} + V_{ou})\, e^{-U_o}. \end{aligned} \tag{18.4}$$

Since H is purely imaginary, the new solution may also be described in terms of the potentials

$$\Psi = \Psi_o \left(\cos^2\alpha + \sin^2\alpha\, e^{-2U_o}\Psi_o^{-1}\right)^2, \qquad \Phi = 0. \tag{18.5}$$

Hence the potential Z may be determined using (16.19).

In his original paper, Panov (1978) applied this transformation to the Khan–Penrose solution. In his second paper (1979*a*), he applied it to the general solution given by Szekeres (1972) as described in Section 14.2. In these solutions there is always a space-like curvature singularity on the surface given by $e^{-U} = f + g = 0$ on which the opposing waves mutually focus each other.

The technique described in this section will be referred to again in the following section. It will be shown there how other solutions that do not contain strong curvature singularities, such as the Bell–Szekeres solution, may be included in the notation of this section.

18.2 An alternative approach

The class of solutions in which the metric can be diagonalized is characterized by the fact that the Ernst potentials Z and E are real and that $\omega = 0$. In terms of the functions of the Szekeres line element (6.20), the equivalent condition is that $W = 0$, and hence from (6.22*e*), that $\Phi_0\bar{\Phi}_2 = \Phi_2\bar{\Phi}_0$. In the general case, the components Φ_0, Φ_2 and H or η must remain complex.

In this case the main field equations (16.6) putting $\chi = e^{-V}$, or (6.22 d, e) expressed in terms of the potential H, together with Maxwell's equations (16.7), can be written in the form

$$\begin{aligned} 2V_{uv} - U_u V_v - U_v V_u + 2e^{U-V}(H_u \bar{H}_v + \bar{H}_u H_v) &= 0 \\ H_u \bar{H}_v - \bar{H}_u H_v &= 0 \\ 2H_{uv} - V_u H_v - V_v H_u &= 0 \end{aligned} \tag{18.6}$$

In the particular case when the polarization vectors of the approaching gravitational waves and electromagnetic waves are all aligned, it is possible to use a duality transformation to ensure that Φ_0, Φ_2 and thus H or η are all real.

In order to generate exact solutions of this type, it is appropriate to consider the technique described at the end of Section 17.4. Accordingly, we consider an initial solution of the vacuum field equations describing the collision of plane gravitational waves. This may be described in terms of the initial Ernst potential E_o which is now real. The potentials for a new diagonal solution describing the collision of plane electromagnetic waves, usually coupled to gravitational waves, are then given by (17.32) or (17.34), where a, b, c and d are all real and satisfy (17.33). Such solutions clearly revert to the original vacuum solution when $a = 1$, $b = 0$, $c = 0$ and $d = 0$.

It is of interest initially to consider the restricted class of solutions that can be obtained from (17.32) when $b = 0$ and $d = 0$. In this case $c = \sqrt{1 - a^2}$ and the transformation reduces to (17.24). Then, reverting to the alternative parameter α given by (17.25), the new solution is given partly as in (17.27) by

$$\begin{aligned} &Z = \frac{\sin^2\alpha + \cos^2\alpha\, Z_\text{o}}{\cos^2\alpha + \sin^2\alpha\, Z_\text{o}} \qquad H = \frac{\sin\alpha\cos\alpha\,(Z_\text{o} - 1)}{\cos^2\alpha + \sin^2\alpha\, Z_\text{o}} \\ &\Psi = Z - H^2 = Z_\text{o}(\cos^2\alpha + \sin^2\alpha\, Z_\text{o})^{-2} \end{aligned} \tag{18.7}$$

The metric functions V or χ are thus given by

$$e^{-V} = \chi = e^{-V_\text{o}}\left(\cos^2\alpha + \sin^2\alpha\, e^{-(U_\text{o} - V_\text{o})}\right)^2. \tag{18.8}$$

This expression can be seen to be indentical to that of (18.1) apart from an insignificant change in the sign of V. It can therefore be seen that the technique described by (17.24) in the diagonal case is identical to that of Panov (1978, 1979a) that has already been described in the previous section. The complete solution in this case is given exactly as in (18.1) and (18.2) but with a change of sign for V and V_o.

It has been noted in the previous section that solutions obtained using the general technique (17.32) automatically satisfy the boundary conditions for colliding electromagnetic and gravitational waves if those conditions are satisfied for the initial vacuum solution. In the particular case of Panov, as described in Section 18.1, this has already been noted.

It is important to notice that we are here using the Ernst potential rather than the metric functions. Thus using (11.3), (11.11) and (12.36), the functions associated with the initial solution are related by

$$\frac{1+E_\mathrm{o}}{1-E_\mathrm{o}} = Z_\mathrm{o} = \Psi_\mathrm{o} = \frac{\sqrt{1-t^2}\sqrt{1-z^2}}{\chi_\mathrm{o}} = e^{-U_\mathrm{o}}e^{V_\mathrm{o}}. \tag{18.9}$$

Using these identities, the main field equations may be seen to reduce to the single real linear equation

$$\big((1-t^2)V_{\mathrm{o},t}\big)_{,t} - \big((1-z^2)V_{\mathrm{o},z}\big)_{,z} = 0 \tag{18.10}$$

which may be seen to be identical to (10.13), which is the main equation for vacuum solutions with aligned polarization. A general solution of this equation will contain the terms (10.16), (10.20) and (10.22), and can be written in the form

$$\begin{aligned} V_\mathrm{o} = &\sum_n \big[a_n P_n(t)P_n(z) + q_n Q_n(t)P_n(z) + p_n P_n(t)Q_n(z) \\ &+ b_n Q_n(t)Q_n(z)\big] - \tfrac{1}{2}c\log(1-t^2) - \tfrac{1}{2}c\log(1-z^2) \\ &+ \sum_i d_i \cosh^{-1}\left(\frac{c_i - tz}{\sqrt{1-t^2}\sqrt{1-z^2}}\right) \end{aligned} \tag{18.11}$$

where $P_n(x)$ and $Q_n(x)$ are Legendre functions of the first and second kinds respectively, and a_n, p_n, q_n, b_n c_i and d_i are sets of arbitrary constants.

In his first paper, Panov (1978) derived a new solution using the above technique with $b=0$ and $d=0$ and with the initial solution taken to be the Khan–Penrose solution given as in (10.18) by

$$V_\mathrm{o} = -2Q_0(t)P_0(z). \tag{18.12}$$

In his second paper (1979*a*), he applied the same technique to the general solution given by Szekeres (1972).

It may also be observed using (16.31) that the Bell–Szekeres solution can be obtained with this method by setting $a=0$, $b=0$, $c=1$ and $d=0$ and using the initial solution of (18.8) given by

$$\begin{aligned} V_\mathrm{o} &= -2P_0(t)Q_0(z) - \tfrac{1}{2}\log(1-t^2) - \tfrac{1}{2}\log(1-z^2) \\ &= -2P_0(t)Q_0(z) + U_\mathrm{o}. \end{aligned} \tag{18.13}$$

However, as noted in (16.32,33), the same solution is also obtained if the term $2P_0(t)Q_0(z)$ is replaced by $2Q_0(t)P_0(z)$. It can then be seen from (10.35) that the initial solution, in this case, is the degenerate Ferrari–Ibañez solution described in Section 10.5.

In their substantial paper that has already been frequently quoted, Chandrasekhar and Xanthopoulos (1987*a*) have also described a general class of diagonal solutions. This has also been obtained using the above approach but with the constraint that $a=0$, $b=0$, $c=1$ and $d=0$ so that $Z=1$. In this case, it is convenient to introduce a new function F such that

$$F = \tfrac{1}{2}(V_{\rm o} - U_{\rm o}). \tag{18.14}$$

With this, the new solution can be characterized by the potentials

$$Z = 1, \qquad H = \tanh F \tag{18.15}$$

and the metric and field components can conveniently be expressed in the form

$$\begin{aligned} U &= U_{\rm o}, & e^{-V} &= e^{-U}\cosh^2 F, \\ W &= 0, & e^{-M} &= e^{2F}\cosh^2 F e^{-M_{\rm o}}, \\ \Phi_0^\circ &= -\operatorname{sech} F F_v, & \Phi_2^\circ &= \operatorname{sech} F F_u. \end{aligned} \tag{18.16}$$

Chandrasekhar and Xanthopoulos (1987*a*) have also obtained the interesting result that if, using the transformation (18.16), the initial solution is chosen such that

$$F = \tfrac{1}{2}\sum_{n=0}^{\infty} a_n P_n(t)P_n(z) - Q_0(t)P_0(z) \tag{18.17}$$

then, in the new solution, the surface on which $t=1$ does not correspond to a curvature singularity as in the Bell–Szekeres solution and the solution of Section 17.3.

Using the same definition (18.14), it is also possible to show that the more general solution obtained from (17.32) can be written in the form

$$\begin{aligned} U &= U_{\rm o}, & e^{-V} &= \frac{1}{(a^2+c^2)} e^{-U_{\rm o}}\big((1-b)\cosh F - a\sinh F\big)^2, \\ W &= 0, & e^{-M} &= e^{2F}\big((1-b)\cosh F - a\sinh F\big)^2 e^{-M_{\rm o}}, \\ \Phi_0^\circ &= -\frac{(c+ad-bc)F_v}{(1-b)\cosh F - a\sinh F}, & \Phi_2^\circ &= \frac{(c+ad-bc)F_v}{(1-b)\cosh F - a\sinh F}. \end{aligned} \tag{18.18}$$

It may be noted that a very general solution may be obtained by taking the initial solution in the form (18.11), and using this generation technique given by (18.18) with F given by (18.14). It would appear that such general solutions would normally contain a scalar curvature singularity on the surface $t = 1$. However, although this class of solutions seems to be very general, it does not necessarily contain the complete class of all possible diagonal solutions, even for the restricted case in which Φ_0, Φ_2 and thus H or η are all real.

18.3 Electromagnetic Gowdy cosmologies

It has already been pointed out in Section 10.7 that the Gowdy cosmologies, which contain interacting plane waves, can under certain conditions also be made to satisfy the necessary boundary conditions appropriate for their interpretation in terms of a collision of initial plane waves. The extension to the electromagnetic Gowdy cosmologies has been carried out by Charach (1979). A reinterpretation of these solutions as the interaction following the collision of plane electromagnetic waves is now presented in the above notation.

It is first appropriate to re-introduce the alternative coordinate system (10.58) defined by

$$\tilde{t} = f + g, \qquad \tilde{z} = f - g \tag{18.19}$$

where $\tilde{t}$ is a past pointing, time-like coordinate which decreases towards the singularity when $\tilde{t} = 0$ in region IV. With this, the line element for the diagonal metric is given by (10.59), and the main field equations (18.6) take the form

$$\ddot{V} + \frac{1}{\tilde{t}}\dot{V} - V'' + \frac{2}{\tilde{t}}e^{-V}(\dot{H}\dot{\bar{H}} - H'\bar{H}') = 0 \tag{18.20a}$$

$$\dot{H}\bar{H}' - H'\dot{\bar{H}} = 0 \tag{18.20b}$$

$$\ddot{H} - H'' - \dot{V}\dot{H} + V'H' = 0 \tag{18.20c}$$

where the dot and prime denote derivatives with respect to $\tilde{t}$ and $\tilde{z}$ respectively.

To express these equations in the form of Carach (1979), it is convenient first to put

$$V = 2\psi - \log\tilde{t}, \qquad \text{or} \qquad \psi = \tfrac{1}{2}(V - U). \tag{18.21}$$

It is then possible to put

$$\dot{H} = -\dot{\rho} + \frac{i}{\tilde{t}}e^{2\psi}\sigma', \qquad H' = -\rho' + \frac{i}{\tilde{t}}e^{2\psi}\dot{\sigma} \tag{18.22}$$

where ρ and σ are the two non-zero components of the four-potential for the electromagnetic field. With them the non-zero components of the electromagnetic field tensor are given by

$$\begin{aligned} F_{14} &= -F_{41} = \dot\sigma, \qquad & F_{24} &= -F_{42} = \sigma', \\ F_{13} &= -F_{31} = \dot\rho, \qquad & F_{23} &= -F_{32} = \rho'. \end{aligned} \tag{18.23}$$

With these substitutions, the main field equations together with Maxwell's equations take the form

$$\begin{aligned} \ddot\psi + \frac{1}{\tilde t}\dot\psi - \psi'' &= -e^{-2\psi}(\dot\rho^2 - {\rho'}^2) + \frac{1}{\tilde t^2} e^{2\psi}(\dot\sigma^2 - {\sigma'}^2) \\ \dot\rho\dot\sigma - \rho'\sigma' &= 0 \\ \ddot\rho + \frac{1}{\tilde t}\dot\rho - \rho'' &= 2(\dot\psi\dot\rho - \psi'\rho') \\ \ddot\sigma - \frac{1}{\tilde t}\dot\sigma - \sigma'' &= 2(\dot\psi\dot\sigma - \psi'\sigma'). \end{aligned} \tag{18.24}$$

It may be noticed that the imaginary part of (18.20c) is automatically satisfied, and that the fourth equation in (18.24) enters as the integrability condition for the definitions (18.22).

Charach has presented a family of exact solutions of these equations for which ψ and ρ are functionally related by the equation

$$\psi = \tfrac{1}{2}\log P(\rho), \qquad \text{and} \qquad \sigma = 0. \tag{18.25}$$

In this case, $H = -\rho$, and the equations (18.24) become

$$\begin{aligned} \ddot\rho + \frac{1}{\tilde t}\dot\rho - \rho'' - \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}\rho}(\dot\rho^2 - {\rho'}^2) &= 0 \\ \left(\frac{\mathrm{d}^2 P}{\mathrm{d}\rho^2} + 2\right)(\dot\rho^2 - {\rho'}^2) &= 0. \end{aligned} \tag{18.26}$$

It may be assumed that $\dot\rho \neq \pm\rho'$ as this does not lead to solutions with interacting waves. It follows then that P can be written in the form

$$P = c_1 + c_2\rho - \rho^2 \tag{18.27}$$

where c_1 and c_2 are arbitrary constants. The non-linearity in the resulting equation can then be removed by the substitution

$$\rho = \rho_\mathrm{o} - a\tanh aF \tag{18.28}$$

provided

$$c_1 = a^2 - \rho_o^2, \qquad c_2 = 2\rho_o \tag{18.29}$$

and F is a solution of the linear equation

$$\ddot{F} + \frac{1}{\tilde{t}}\dot{F} - F'' = 0. \tag{18.30}$$

It may immediately be noticed that (18.30) is identical to the vacuum equation (10.60) as may also be deduced from (18.20a). This equation may also be written in terms of different coordinates in the form of the vacuum equations (10.2) or (9.3) as

$$2F_{uv} - U_u F_v - U_v F_u = 0. \tag{18.31}$$

The resulting metric function is then given by $e^{-V} = e^{-U} a^{-2} \cosh^2 aF$.

The method described above can clearly be used to generate electromagnetic solutions from any initial vacuum solution. It may be noticed, however, that the constant ρ_o can be removed from the electromagnetic potential, and that it is possible to set $a = 1$ without loss of generality. In this case the solutions are characterized by the potentials

$$Z = 1, \qquad H = \tanh F \tag{18.32}$$

which can be seen to be identical to (18.15). This solution-generating technique is thus identical to that of Chandrasekhar and Xanthopoulos (1987a) as described at the end of the previous section. In terms of the function F, the metric and field components are given by (18.16).

Charach (1979) has presented the class of solutions in which

$$\begin{aligned} F = &-a \log \tilde{t} + \sum_{\omega} \{A_\omega \cos[\omega(\tilde{z} + \alpha_\omega)]\, J_0(\omega\tilde{t})\} \\ &+ \sum_{\omega} \{B_\omega \cos[\omega(\tilde{z} + \beta_\omega)] Y_0(\omega\tilde{t})\} \end{aligned} \tag{18.33}$$

where $J_0(n\tilde{t})$ and $Y_0(n\tilde{t})$ are Bessel functions of the first and second kinds of zero order, and A_n, B_n, α_n and β_n are sets of arbitrary constants. This has been obtained by separating the variables in (18.30), and considering only solutions that are periodic in $\tilde{z}$.

As pointed out by Feinstein and Ibañez (1989) and described here in Section 10.7, these solutions can easily be adapted to the boundary conditions for colliding plane waves if F also includes the terms

$$F = \sum_n d_n \cosh^{-1}\left(\frac{\tilde{z} + c_n}{\tilde{t}}\right). \tag{18.34}$$

It may also be noted that, since the equation (18.30) or (18.31) for F is the by now familiar Euler–Poisson–Darboux equation, further explicit solutions can be obtained using any combination of the terms described previously in Section 10.8.

18.4 Other methods

Further diagonal solutions which describe colliding plane electromagnetic waves can easily be generated from known solutions, and also from known colliding plane gravitational wave solutions, using the invariance transformations (17.35*a*,*d*,*e*) with real constants. These transformations are generalizations of those such as (18.7) that have been considered above. Other methods are also available.

Within the context of stationary axisymmetric solutions, it is well known that a Bonnor transformation (Bonnor, 1961) can be used to relate a class of diagonal electromagnetic solutions to a class of non-diagonal vacuum solutions. The precise result may be stated in the form that if Z_o is a complex solution of the vacuum Ernst equation (11.8), then a solution of the Ernst equations (16.22) for an electromagnetic field is given by

$$Z = Z_\mathrm{o}\bar{Z}_\mathrm{o}, \qquad H = \tfrac{1}{2}(\bar{Z}_\mathrm{o} - Z_\mathrm{o}) \tag{18.35}$$

where it may be noted that here H is purely imaginary. The inverse transformation also holds.

It was first suggested by Halilsoy (1983) that this transformation could be used to generate colliding plane wave solutions. However, there is considerable difficulty with the application of the boundary conditions. If Z_o satisfies the junction conditions (11.26) for colliding non-aligned gravitational plane waves, then (Z, H) given by (18.35) do not necessarily satisfy the junction conditions (16.28) for colliding aligned electromagnetic waves. All is not lost, however, since it is possible to start with solutions of (11.8) or (16.22) which can not be interpreted as colliding plane waves, and then to use the transformation (18.35) or its inverse to generate solutions which may be so interpreted for certain values of the free parameters.

In view of the large number of solutions of Ernst's equation that are now known, such an approach to the generation of colliding plane wave solutions is feasible. However, it has not yet been used effectively.

19

ELECTROMAGNETIC WAVES COLLIDING WITH GRAVITATIONAL WAVES

In the previous chapters we have considered the collision between two electromagnetic waves, permitting them also to be coupled to gravitational waves. In this chapter a situation is considered in which a plane gravitational wave approaches and collides with a plane electromagnetic wave. In this situation, the field equations are identical to those of previous chapters but the initial conditions are different.

19.1 A simple example

A particularly simple example is the case when an impulsive gravitational wave described by the component $\Psi_4 = a\delta(u)$ collides with a step electromagnetic wave described by $\Phi_0 = b\Theta(v)$. An exact solution describing this case has been given by the author (Griffiths 1975*b*). In this case, the line elements describing the initial regions are given by

$$\begin{aligned}
&\text{Region I}: && \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \\
&\text{Region II}: && \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1-au)^2\mathrm{d}x^2 - (1+au)^2\mathrm{d}y^2 \\
&\text{Region III}: && \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 bv(\mathrm{d}x^2 + \mathrm{d}y^2).
\end{aligned} \tag{19.1}$$

In terms of the metric functions of the Szekeres line element (6.20), the solution of the field equations (6.21-22) for the interaction region IV, satisfying the required boundary conditions is given by

$$\begin{aligned}
e^{-U} &= \cos^2 bv - a^2u^2, & e^V &= \frac{(1-au)}{(1+au)}, \\
e^{-M} &= \frac{\cos bv\sqrt{1-a^2u^2}}{\sqrt{\cos^2 bv - a^2u^2}}, & W &= 0, \\
\Phi_0^\circ &= \frac{b\cos bv}{\sqrt{\cos^2 bv - a^2u^2}}, & \Phi_2^\circ &= -\frac{a\sin bv}{(1-a^2u^2)\sqrt{\cos^2 bv - a^2u^2}}.
\end{aligned} \tag{19.2}$$

This solution can be seen to have a number of interesting features. The incoming gravitational and electromagnetic waves are plane and parallel propagating. That is, they follow non-expanding twist-free and

shear-free null geodesic congruences. When the gravitational wave meets the electromagnetic wave it starts to contract. By contrast, the electromagnetic wave both contracts and shears. The contraction of both waves becomes unbounded on the space-like hypersurface $\cos^2 bv = a^2u^2$ which can be seen to correspond to a scalar polynomial curvature singularity. These properties are as expected according to the discussion in Chapter 5.

It can be seen from (19.2) that the electromagnetic wave is partially reflected on collision with the gravitational wave. This occurs because the congruence along which the electromagnetic wave propagates is focused astigmatically by the gravitational wave, and is no longer shear-free. According to the Mariot–Robinson theorem 5.2, null electromagnetic waves necessarily propagate along shear-free null geodesics. It follows that the electromagnetic field in the interaction region cannot remain null and therefore some back scattering must occur.

This particular feature has also been confirmed by Sbytov (1973), who considered the propagation of a test electromagnetic wave through a plane gravitational wave.

Other interesting features of this solution can be seen after first evaluating the gravitational wave components. Using (6.23), the only non-zero components in the interaction region are given by

$$\begin{aligned}\Psi_4^\circ &= a\delta(u) - \frac{3a^3u\sin^2 bv}{(1-a^2u^2)^2(\cos^2 bv - a^2u^2)}\\ \Psi_2^\circ &= -\frac{a^2b\,u\sin bv\cos bv}{(\cos^2 bv - a^2u^2)^2}.\end{aligned} \tag{19.3}$$

From the absence of a Ψ_0 component, it can be seen that, unlike the electromagnetic wave, the gravitational wave is not partially reflected. However, the usual coulomb component Ψ_2 is still generated by the collision, and the component Ψ_4 develops a tail.

It follows from the absence of the Ψ_0 and Ψ_1 terms that the gravitational field in the interaction region, as represented by the Weyl tensor, is of algebraic type II. However, in this situation, the repeated principal null congruence of the gravitational field is not aligned with a principal null congruence of the electromagnetic field.

Another simple solution has been given elsewhere (Griffiths 1976*b*). This has the same general properties as the above solution, but the initial impulsive gravitational wave has been replaced by a step wave with the line element (4.21).

19.2 General initial data

Consider now the situation of a totally general collision between an arbitrary gravitational wave and an arbitrary electromagnetic wave. It is

convenient to assume that the gravitational wave approaches in region II, and the electromagnetic wave in region III. The background in region I is assumed to be flat with metric (3.6).

Region II is now considered to contain a general gravitational wave. The appropriate line element is (4.13). This may be compared with the Szekeres line element (6.20) with the condition that $V = V(u)$ and $W = W(u)$. Also $e^{-U} = (\frac{1}{2} - f)$ where $f = f(u)$, and it is possible to scale the coordinate u such that $e^{-M} = 1$ giving

$$ds^2 = 2\mathrm{d}u\mathrm{d}v - (\tfrac{1}{2}+f)\big(e^V \cosh W \mathrm{d}x^2 - 2\sinh W \mathrm{d}x\mathrm{d}y + e^{-V}\cosh W \mathrm{d}y^2\big). \tag{19.4}$$

This gravitational wave is completely arbitrary and is described in terms of three functions $f(u)$, $V(u)$ and $W(u)$ that are constrained only by the single equation

$$\frac{2f''}{(\frac{1}{2}+f)} - \frac{f'^2}{(\frac{1}{2}+f)^2} + W'^2 + V'^2\cosh^2 W = 0 \tag{19.5}$$

which is equivalent to (6.22c). From (6.23) it can be seen that the approaching gravitational wave is given by the single component

$$\begin{aligned}\Psi_4(\textsc{ii}) = &-\tfrac{1}{2}(V''\cosh W + iW'') - V'W'\sinh W + \tfrac{1}{2}iV'^2\sinh W\cosh W \\ &- \frac{f'}{2(\frac{1}{2}+f)}(V'\cosh W + iW').\end{aligned} \tag{19.6}$$

In order to satisfy the appropriate boundary conditions across the wave front, it is also necessary to assume that

$$f = \tfrac{1}{2}, \qquad V = W = f' = V' = W' = 0, \qquad \text{when} \qquad u = 0. \tag{19.7}$$

Region III here may be considered to contain a general plane electromagnetic wave. It is assumed that there is no associated free gravitational wave, so that the metric is asymptotically flat. It turns out to be convenient to take the line element in this region in the form

$$ds^2 = 2e^{-M}\mathrm{d}u\mathrm{d}v - (\tfrac{1}{2}+g)(\mathrm{d}x^2+\mathrm{d}y^2) \tag{19.8}$$

where $g = g(v)$ and $M = M(v)$. The electromagnetic wave is described by the component Φ_0. The amplitude of this wave is determined by the equation (6.22b) which becomes

$$4\Phi_0^\circ\bar\Phi_0^\circ = -\frac{2g''}{(\frac{1}{2}+g)} + \frac{g'^2}{(\frac{1}{2}+g)^2} - \frac{2g'M'}{(\frac{1}{2}+g)}. \tag{19.9}$$

The phase of the wave, however, is completely arbitrary, as is required by the fact that a plane electromagnetic wave is only determined by the metric up to an arbitrary duality rotation.

It is, of course, possible to rescale the null coordinate v in order to put $M = 0$. This would leave the electromagnetic wave being described by the single function $g(u)$ and an arbitrary phase. In order to ease the integration of the field equations in region IV, however, it is preferable to retain the additional function, and to regard $g(v)$, $\Phi_0(v)$ and $M(v)$ as arbitrary functions that, in region III, are required to satisfy (19.9).

In order to satisfy the junction conditions across the boundary between regions I and III, it is assumed that

$$g = \tfrac{1}{2}, \qquad M = g' = M' = 0, \qquad \text{when} \qquad v = 0. \tag{19.10}$$

The initial conditions that have now been set describe the collision between a completely general gravitational wave and a completely general pure electromagnetic wave. The corresponding general solution describing the subsequent interaction has not yet been obtained. However, one particular class can easily be obtained and will be described in the next section.

19.3 A general class of solutions

As frequently described in previous chapters, the line element in the interaction region may be considered in the Szekeres form (6.20), and the field equations are (6.21) and (6.22).

As in (6.24), it is always possible to integrate (6.22a) to obtain

$$e^{-U} = f(u) + g(v) \tag{19.11}$$

where, to satisfy the boundary conditions, $f(u)$ and $g(v)$ necessarily take the same form as they have in regions II and III respectively. These functions are thus exactly of the form that is specified by the approaching waves.

A general class of solutions (Griffiths 1983) has previously been obtained in which the metric functions V and W are assumed to be independent of v. In this case

$$V = V(u) \qquad \text{and} \qquad W = W(u) \tag{19.12}$$

take exactly the same form in the interaction region as they do in region II. This class of solutions is thus almost completely determined by the functions $f(u)$, $V(u)$, $W(u)$ and $g(v)$ that are specified by the approaching waves.

With the condition (19.12), the field equations (6.22*d*) and (6.22*e*) imply that

$$\Phi_0^\circ \bar{\Phi}_2^\circ = \frac{g'(V'\cosh W - iW')}{4(f+g)}. \tag{19.13}$$

With this condition, Maxwell's equations (6.21) then imply that

$$\begin{aligned} \Phi_0^\circ &= \frac{-g'}{2\sqrt{f+g}\sqrt{\frac{1}{2}-g}}\, e^{i\alpha(u)} \\ \Phi_2^\circ &= -\frac{\sqrt{\frac{1}{2}-g}}{2\sqrt{f+g}}(V'\cosh W + iW')\, e^{i\alpha(u)} \end{aligned} \tag{19.14}$$

where α is now a function of u only, and is required to satisfy the equation

$$\alpha' = -\tfrac{1}{2} V' \sinh W. \tag{19.15}$$

The electromagnetic field is now determined completely by the given functions up to an arbitrary constant phase. Finally, the remaining equations in (6.22) can be integrated to give

$$e^{-M} = \frac{-g'\sqrt{\frac{1}{2}+f}}{2a\sqrt{f+g}\sqrt{\frac{1}{2}-g}} \tag{19.16}$$

where a is an arbitrary constant.

The metric functions and the electromagnetic field components in the interaction region are now all determined. The components of the Weyl tensor can be obtained from (6.23) and, using (19.6), the non-zero components can conveniently be written in the form

$$\begin{aligned} \Psi_4^\circ(\text{IV}) &= \Psi_4(\text{II}) - \frac{3f'(\frac{1}{2}-g)}{4(f+g)(\frac{1}{2}+f)}(V'\cosh W + iW') \\ \Psi_2^\circ(\text{IV}) &= -\frac{f'g'}{4(f+g)^2}. \end{aligned} \tag{19.17}$$

Two fundamental properties of this solution can immediately be deduced from these expressions.

Firstly, it may be observed that these solutions necessarily contain a scalar polynomial curvature singularity in region IV on the space-like surface given by $f+g=0$. This is common to most colliding plane wave solutions.

Secondly, it may be observed that the gravitational wave is algebraically special and is of algebraic type II. The congruence on which $v = \text{const}$ is a repeated principal null congruence of the gravitational field, but is not a principal null congruence of the electromagnetic field.

This class of algebraically special solutions thus contains a non-aligned non-null electromagnetic field. Its relation with other non-aligned non-null Einstein–Maxwell fields has been investigated elsewhere (Griffiths 1986), where a more general class of such fields has been obtained. This more general class contains two distinct sub-classes, one of which is the class obtained here, while the other contains all other known solutions of this type.

Another important property of this class of solutions can be deduced from the electromagnetic field components given by (19.14). It can immediately be seen that the electromagnetic wave has been partially reflected by the gravitational wave. This property has already been noted in the special case described in Section 19.1.

It may also be noticed that the expression for M given by (19.16) is only continuous across the boundary between regions III and IV if, in region III, M is given by

$$e^{-M} = \frac{-g'}{2a\sqrt{\frac{1}{2}-g}\sqrt{\frac{1}{2}+g}}. \tag{19.18}$$

where a is a constant. The other boundary conditions (19.10) are then only satisfied if g has the form

$$g = \tfrac{1}{2} - a^2v^2 + \ldots \tag{19.19}$$

The approaching electromagnetic wave is then given by

$$\Phi_0^\circ(\text{III}) = \frac{-g'}{2\sqrt{\frac{1}{2}-g}\sqrt{\frac{1}{2}+g}}e^{i\alpha} \tag{19.20}$$

where α is a constant. In this case, a coordinate transformation

$$v \to \tilde{v}, \qquad \text{where} \qquad g = \tfrac{1}{2} - \sin^2 a\tilde{v} \tag{19.21}$$

can be used to put $M = 0$, and it can then be seen that the approaching wave is necessarily the step wave given by

$$\Phi_0(\text{III}) = ae^{i\alpha}\Theta(\tilde{v}) \tag{19.22}$$

where the arbitrary constant phase is of no significance.

Since M and g are now completely determined with $M = 1$ in region III, it follows that the electromagnetic field component is also completely determined. Thus, although this solution includes a completely general gravitational wave in region II, the approaching electromagnetic wave can only be the step wave given by (19.22).

Finally, it is of interest to re-express this class of solution in terms of the notation of the previous chapters. With the components (19.14), equations (16.4) can be integrated to give the potential function H in the form

$$H = \sqrt{\tfrac{1}{2} - g}\, e^{V/2} \sqrt{1 + i \sinh W}\, e^{i\alpha}. \tag{19.23}$$

In view of (19.12) and the definitions (11.2) and (11.3), it is clear that χ and ω are here functions of u only. However, they may subsequently be considered as functions of f only or, using (10.9) (see also the appendix), as functions of $(\psi + \lambda)$ only:

$$\chi = \chi(\psi + \lambda), \qquad \omega = \omega(\psi + \lambda). \tag{19.24}$$

This approach, however, does not lead to simple expressions for Ψ, Φ and Z defined respectively by (16.14), (16.13) and (16.19).

20

OTHER SOURCES

So far only solutions of the Einstein or Einstein-Maxwell equations have been considered. This has enabled the properties of colliding plane waves to be considered in the various cases in which the approaching waves were gravitational waves, pure electromagnetic waves, or combinations of both. A number of similar results have also been obtained for a variety of other sources. These will be reviewed in this chapter.

20.1 Scalar fields

A collision between two complex, massless, scalar plane waves has been considered by Wu (1982). A consideration of such a situation is motivated by its approximation to interactions that may be considered to have occurred in the early universe.

Such a field is given by a scalar field ϕ satisfying

$$\phi^{;\mu}{}_{;\mu} = 0 \tag{20.1}$$

with energy-momentum tensor given by

$$T_{\mu\nu} = \frac{1}{8\pi}(\phi_{,\mu}\bar{\phi}_{,\nu} + \phi_{,\nu}\bar{\phi}_{,\mu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\bar{\phi}^{,\alpha}). \tag{20.2}$$

Thus the only non-zero components of the Ricci tensor are

$$\begin{aligned} \Phi_{00} &= \phi_v\bar{\phi}_v, \qquad \Phi_{22} = \phi_u\bar{\phi}_u, \\ \Phi_{11} &= -\Lambda/3 = \tfrac{1}{4}(\phi_v\bar{\phi}_u + \phi_u\bar{\phi}_v). \end{aligned} \tag{20.3}$$

For colliding scalar plane waves it may be assumed that ϕ is a constant that may be taken to be zero in region I, that it is a function of u in region II, that it is a function of v in region III, and that it depends on both u and v in the interaction region IV. Using the Szekeres line element (6.20) it can be shown that, if the background region I is flat and if regions II and III do not contain independent gravitational waves with components Ψ_4 and Ψ_0, then it is possible to put $V=0$ and $W=0$ everywhere. The scalar wave equation (20.1) can then be written in the form

$$2\phi_{uv} = U_u\phi_v + U_v\phi_u \tag{20.4}$$

and the remaining field equations are

$$\begin{aligned} U_{uv} &= U_u U_v \\ 2U_{vv} &= U_v^2 - 2U_v M_v + 4\phi_v\bar{\phi}_v \\ 2U_{uu} &= U_u^2 - 2U_u M_u + 4\phi_u\bar{\phi}_u \\ 2M_{uv} &= -U_u U_v + 2(\phi_u\bar{\phi}_v + \phi_v\bar{\phi}_u). \end{aligned} \tag{20.5}$$

It is perhaps worth pointing out that the above equations have also been given by Tabensky and Taub (1973) in their study of plane symmetric self-gravitating fluids in which the pressure is equal to the density. They have pointed out that such 'stiff fluids' also have a dual interpretation in terms of scalar fields.

In this context, however, it may immediately be noticed that the scalar wave equation (20.4) is identical in form to (9.3), which is the main equation for colliding aligned gravitational waves, but with ϕ replacing V. Moreover, when ϕ is real, the remaining equations (20.5) are also of identical form to the remaining equations for colliding colinear gravitational waves as given by (6.21) with $W = 0$, $\Phi_0^\circ = \Phi_0^\circ = 0$ and V replaced by 2ϕ. It can thus be seen that, if U_o, V_o and M_o with $W_\mathrm{o} = 0$ is a solution of the vacuum equations describing a colinear collision of gravitational waves, then a solution of the above equations describing a collision of scalar plane waves is given by

$$U = U_\mathrm{o}, \qquad M = M_\mathrm{o}, \qquad \phi = \tfrac{1}{2}V_\mathrm{o}e^{i\alpha} + \phi_\mathrm{o} \tag{20.6}$$

where α and ϕ_o are arbitrary constants with α real. Moreover, the appropriate junction conditions are automatically satisfied if they are satisfied for the initial solution.

Wu (1982) has given a solution of the above type using the expression for V_o taken from the Szekeres solution (9.4). Halilsoy (1985) has further generalized this using the transformation (12.1). In fact, since the equation (20.4) is linear, the general solution can be expressed as any of the infinite series (10.78–82). Alternatively any finite combination of the terms contained in these series may be used, as described in Chapter 10.

A number of explicit solutions of the above equations have also been given by Carmeli, Charach and Feinstein (1983) and interpreted in terms of Gowdy cosmologies containing scalar fields or 'stiff fluids'. Clearly these can easily be adapted to colliding wave solutions using the techniques discussed in Sections 10.7 and 18.3. In addition, the algorithm of Wainwright, Ince and Marshman (1979) can also be used to generate further solutions.

Solutions describing the collision of a scalar plane wave with a gravitational wave, an electromagnetic wave and a neutrino wave have also been described by Wu (1982).

20.2 Perfect fluid solutions

Chandrasekhar and Xanthopoulos (1985*b*,*c*,1986*a*) have also attempted to solve the field equations in the interaction region IV in the presence of a perfect fluid. Such a situation is clearly of interest as it is important to investigate the collision and interaction of waves in a cosmological background. However, the formulation of this problem inevitably leads to certain difficulties.

It is not difficult to modify the field equations (6.22) so that they include the source terms of a perfect fluid. It may also be expected that methods for solving these equations can be related to those for stationary axisymmetric space-times. Some known cylindrically symmetric solutions, or stationary axisymmetric solutions in regions in which there are two commuting space-like Killing vectors, may also appear as solutions of these equations. However, these solutions will be given as functions of the coordinates t and z, or possibly of ψ and λ, or even of f and g. The problem arises when an attempt is made to extend these solutions to initial approaching waves prior to a collision.

The familiar approach adopted in previous chapters has been simply to express the functions f and g in terms of the null coordinates u and v in the form

$$f = \tfrac{1}{2} - (c_1 u)^{n_1}\Theta(u), \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2}\Theta(v) \tag{20.7}$$

or some transformation of this. With this approach, the metric functions in regions II and III are automatically only functions of u and v respectively, and region I is inevitably flat.

In this case, however, the fluid is also described by functions of either of the null coordinates u and v in regions II and III respectively, and so the fluid in these regions is necessarily null. The background region I is flat Minkowski space. Thus the global structure of these solutions is inevitably one in which two null plane fluids approach in a flat background. The fluid subsequent to the collision is non-null.

The interpretation of these solutions as the result of a collision of two null fluids rather than as a collision of waves in a cosmological background inevitably leads to a questioning of the physical significance of these solutions. However, some interesting results have been obtained, and some solution-generating techniques have been developed.

The energy momentum tensor for a perfect fluid is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{20.8}$$

where ρ denotes the energy density and p the pressure. The velocity of the fluid is u^μ, and the conservation equations, which must also be imposed, are given by $T^{\mu\nu}{}_{;\nu} = 0$.

With the assumption of plane symmetry, it is possible to express the velocity in terms of the tetrad vectors used in previous chapters in the form

$$u^\mu = \frac{1}{\sqrt{2}}(al^\mu + bn^\mu) \tag{20.9}$$

where the real functions a and b are constrained such that $ab = 1$. It then follows from Einstein's equations that the only non-zero components of the Ricci tensor are given by

$$\begin{gathered}\Phi_{00} = 2\pi(\rho + p)a^2, \qquad \Phi_{11} = \pi(\rho + p), \qquad \Phi_{22} = 2\pi(\rho + p)b^2, \\ \Lambda = \tfrac{1}{3}\pi(\rho - 3p).\end{gathered} \tag{20.10}$$

It is possible to absorb the functions a and b into the scale factors A and B defined by (6.12–14), and so to put $a = 1$ and $b = 1$. However, it may alternatively be appropriate to retain this comparative freedom, and to look for solutions in which $b \to 0$ and $a \to 0$ on the boundaries of region IV with regions II and III respectively, so that the fluid velocity is continuous with that of the null fluids in those regions.

Since Φ_{01} and Φ_{21} are both zero, it is possible here to adopt the Szekeres metric (6.20), and the field equations, previously expressed as (6.22), then take the form

$$\begin{aligned}
U_{uv} &= U_u U_v - 2\Phi^\circ_{11} - 6\Lambda^\circ && (20.11a)\\
2U_{vv} &= {U_v}^2 + {W_v}^2 + {V_v}^2 \cosh^2 W - 2U_v M_v + 4\Phi^\circ_{00} && (20.11b)\\
2U_{uu} &= {U_u}^2 + {W_u}^2 + {V_u}^2 \cosh^2 W - 2U_u M_u + 4\Phi^\circ_{22} && (20.11c)\\
2V_{uv} &= U_u V_v + U_v V_u - 2(V_u W_v + V_v W_u)\tanh W && (20.11d)\\
2W_{uv} &= U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W && (20.11e)\\
2M_{uv} &= -U_u U_v + W_u W_v + V_u V_v \cosh^2 W + 8\Phi^\circ_{11} && (20.11f)
\end{aligned}$$

using the scale-invariant quantities defined in (6.13).

It may immediately be noticed from (20.11d) and (20.11e) that, if the background region I is flat and the approaching null fluids are not coupled to gravitational waves, then it is possible to put $V = 0$ and $W = 0$ everywhere. However, in most of the exact solutions that have

been given, approaching gravitational waves are also included and the solutions reduce to vacuum colliding wave solutions in the limit as the fluid vanishes.

It may also be noticed that (20.11a) is identical to (6.22a) if

$$\Phi_{11} + 3\Lambda = 0, \qquad \text{or} \qquad \rho = p. \tag{20.12}$$

This is the condition for the so-called ‘stiff fluid’ in which the speed of sound is the same as the speed of light. Such fields have been described by Tabensky and Taub (1973), who have also given plane symmetric solutions satisfying the above conditions although their solutions are not interpreted in terms of colliding waves.

In the particular case in which (20.12) is satisfied, (8.10a) can be integrated to give

$$e^{-U} = f(u) + g(v) \tag{20.13}$$

as in previous chapters.

It may also be noticed that, with this choice for U, the equations (20.11d,e), which are the integrability conditions for the remaining equations, are identical to the main vacuum equations for colliding gravitational waves. The main equations here are thus identical to the vacuum Ernst equation in either of the forms (11.8) or (11.14). In this case, it is possible to use any of the vacuum solutions described in Chapters 9 to 13 to generate solutions for colliding null fluids.

This particular, though somewhat unphysical, situation in which there is a stiff fluid in region IV, has been considered in great detail by Chandrasekhar and Xanthopoulos (1985b). They have shown that the fluid velocity components in this case can be expressed in terms of a potential function ϕ which satisfies the basic hyperbolic equation

$$(1-t^2)\phi_{,tt} - (1-z^2)\phi_{,zz} = 0 \tag{20.14}$$

using the coordinates defined in (10.9–12). They have also given an exact solution which is a generalization of the Nutku–Halil solution described in Section 13.1, and which similarly has an essential curvature singularity in region IV. This solution has also been applied to cosmological situations (Chandrasekhar and Xanthopoulos 1985c). It is clear that the curvature singularity that occurs here in region IV can be considered to be generically similar to the time reverse of the initial singularity that occurs in cosmological models.

Solutions of this type in which the approaching gravitational waves have parallel constant polarization have been considered further by Xanthopoulos (1986).

It is also appropriate here to point out that solution-generating techniques for stiff fluid space-times with two Killing vectors have been described by Kitchingham (1984). However, these were initially applied only in a cosmological context and the boundary conditions for colliding plane waves were not included.

20.3 Null fluids and the uniqueness problem

The perfect fluid solutions described in the previous section were formed from the collision of two null fluids. Chandrasekhar and Xanthopoulos (1986*a*) have therefore also considered the situation in which the two null fluids do not interact in region IV. In this case, the energy-momentum tensor in the interaction region takes the form:

$$T^{\mu\nu} = \rho_1 l^\mu l^\nu + \rho_2 n^\mu n^\nu \tag{20.15}$$

and the only non-zero components of the Ricci tensor are given by

$$\Phi_{00} = 4\pi\rho_2, \qquad \Phi_{22} = 4\pi\rho_1. \tag{20.16}$$

It can immediately be seen that exact solutions of (20.11) in this case are almost identical to exact solutions in which region IV contains a stiff fluid. The only difference is a small change in the expression for M. In particular, it is clear that the two cases may be connected to identical solutions for regions II, III and I.

This latter observation, that two different solutions for two distinct types of field have the same prior extension, is very interesting and has caused much confusion. In this situation, it appears that a given set of initial conditions as set up in regions I, II and III does not give rise to a unique solution in region IV.

This author, at least, does not find this ambiguity or lack of uniqueness in any way surprising. It seems to me to follow inevitably from the way in which fluids are treated in general relativity. Perfect fluids, as introduced above, have not been treated in terms of a complete set of field equations. We have not started with a variational principle and then derived field equations and an expression for the energy-momentum tensor. Rather, it has simply been assumed that the energy-momentum tensor takes the form (20.8) or (20.15), and the only other condition which is automatically satisfied is that it be divergence-free. This approach is in marked contrast to that considered above for colliding electromagnetic waves in which Maxwell's equations are required to be satisfied everywhere, in addition to specifying an expression for the energy-momentum

tensor. I would therefore argue that the absence of field equations, and the method of defining a fluid simply by specifying a particular form of the energy-momentum tensor, will inevitably lead to a lack of uniqueness when matching solutions across different regions of space-time.

This apparent ambiguity in the future space-time for the same initial conditions has been further investigated by Taub (1988*a,b*). He has taken the view that, in order to obtain a unique future time development, in addition to the initial metrics it is also necessary to specify the form of the energy-momentum tensor in region IV. Using this approach, he has determined the conditions on the Ricci tensor that ensure that the evolution of such a space-time is uniquely determined.

An alternative approach has been put forward by Feinstein, MacCallum and Senovilla (1989). They argue that the ambiguous evolution of the space-time is not caused primarily by a lack of conditions on the Ricci tensor but, rather, by an incomplete physical treatment of the problem. They have taken the view that the traditional model of a perfect fluid, as given above by (20.8), represents a macrosopic average of the real physical fields that make up the fluid. In considering the interactions of such fields, they argue that it is necessary to consider all the appropriate field equations. By contrast, in the solutions of Chandrasekhar and Xanthopoulos (1985*c*, 1986*a*) and of Taub (1988*a,b*), it is only the gravitational field equations and the continuity equation that have been considered. The ambiguity in the future development following the collision is thus considered to arise from a lack of field equations.

In a similar way, Hayward (1990*a*) has argued that the apparent paradox should be resolved by distingishing between different types of 'null dust' representing different types of matter field.

Feinstein, MacCallum and Senovilla (1989) have also reconsidered the solutions of Chandrasekhar and Xanthopoulos (1985*b*) which contain a stiff fluid in region IV. They have shown that these can be reinterpreted in terms of the collision of two massless scalar mesons which produce a minimally coupled massless scalar field in the interaction region. In this case, the Klein–Gordon equation is satisfied everywhere.

A number of further exact solutions for colliding null fluids have been given by Taub (1988*a*). In these solutions, the approaching fluids are not associated with gravitational wave components Ψ_4 and Ψ_0. His solutions for region IV may be empty, or may contain two non-interacting null fluids with energy-momentum tensor given by (20.15), or may contain a stiff perfect fluid satisfying (20.12). In addition to manifesting the ambiguity discussed above, the metric functions for these solutions are only assumed to be C^0 across the boundaries of each region, and so may also contain distribution valued Ricci tensor components on these boundaries. These

extra components may alternatively be interpreted in terms of thin shells of null matter. Such shells are referred to frequently in the literature and will be discussed further in the next section.

Taub has considered this whole question further in a subsequent paper (Taub 1990). In this paper, he has considered the collision of two plane impulsive gravitational waves followed by null dust. He has not initially specified any field equations or the form of the energy-momentum tensor in the interaction region. He has simply imposed the usual continuity conditions across the boundaries. In this way he has obtained the interesting result that the energy-momentum tensor has to take one of three forms. It can either (a) be of the form (20.15) for two non-interacting null fluids, or (b) be of the form for a stiff perfect fluid as considered in Section 20.2, or (c) be of the form for an anisotropic fluid. The form (b) is equivalent to a scalar field, and (c) is equivalent to two independent non-interacting scalar fields.

In agreement with this result of Taub's it may be noted that Ferrari and Ibañez (1989*a,b*) had previously presented a four-parameter family of solutions of Einstein's equations for which the source is a non-perfect fluid with an anisotropic distribution of pressure. In this case the parameters have been chosen such that there are no impulsive components in the Riemann tensor and it has been shown that, for this class, a curvature singularity always develops a finite time after the instant of the collision. Since the Klein–Gordon equation for a massless scalar field is satisfied in the interaction region, and since the scalar field satisfies the required junction conditions across the null boundaries, these solutions can alternatively be interpreted in terms of scalar waves.

It has further been shown (Ferrari and Ibañez 1989*a*) that this class of solutions can be transformed to generate a class of solutions for stiff fluids with $p=\rho$ which includes the solution of Chandrasekhar and Xanthopoulos (1985*b*) as a special case.

It must be pointed out, however, that Taub's result is based on the assumption that the space-time in the interaction region can be written as the product of two two-spaces. In this case the metric can be written either in the Szekeres form (6.20) or in the form (11.4). It is only on this assumption that the energy-momentum tensor must be one of the three types mentioned above. Although this condition is satisfied for all the fields considered so far, it is not necessarily always the case. In fact, in Section 20.5, a solution will be presented for colliding neutrino fields in which the metric in the interaction region is not block diagonal and the line element takes a very different form.

The collision of plane clouds of null dust in the absence of approaching gravitational waves has been further considered by Tsoubelis and

Wang (1990). In this paper they have described a number of exact solutions which have different interpretations.

20.4 Plane shells of matter

Following the above two sections, which effectively cover the collision of plane waves composed of null dust, it is appropriate to separately review a number of publications which discuss the collision of thin shells of null matter.

The solution of Stoyanov (1979), which has already been described in Section 10.2, must be included in this discussion since it effectively contains impulsive components of the Ricci tensor along the boundaries connecting the regions I, II, III and IV. These components necessarily appear because it has only been required that the metric is C^0 across these boundaries. In addition, it has been pointed out that the singularity in the interaction region IV can only be avoided if the null matter along these boundaries has negative energy.

A more thorough analysis of the collision of thin plane shells of null matter has been given by Dray and 't Hooft (1986). A situation is considered in which the shells of matter are not coupled to gravitational waves. In this case, no shear is induced and the interaction of the two waves is one of pure focusing. The condition has also been imposed that the two shells pass through each other without interacting and continue along the null surfaces given by $u =$ constant and $v =$ constant. The solution can be described in the present notation by putting $n_1 = n_2 = 1$ in (20.7). This gives the C^0 expressions

$$f = \tfrac{1}{2} - c_1 u\Theta(u), \qquad g = \tfrac{1}{2} - c_2 v\Theta(v) \tag{20.17}$$

in which the positive constants c_1 and c_2 define the energy density of the two shells. Using the Szekeres line element (6.20), the field equations (6.22) can be satisfied by putting $V = W = 0$, $M = \frac{1}{2}\log(f+g)$ and the metric can be written as

$$\mathrm{d}s^2 = \frac{2}{\sqrt{f+g}}\mathrm{d}u\mathrm{d}v - (f+g)(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{20.18}$$

In the interaction region it can be seen that this is a Kasner metric which contains an essential curvature singularity on the space-like surface given by $f(u) + g(v) = 0$.

There is, however, one problem that occurs in situations of this type. As pointed out at the end of the previous section, the solution in the interaction region following a collision is not uniquely determined by the

set of initial data that has been given. Chandrasekhar and Xanthopoulos (1985*c*, 1986*a*) have illustrated this fact by producing two different solutions for the interaction region IV that can develop from the same initial data in regions I, II and III which describe two approaching gravitational waves coupled with null fluids. Taub (1988*a,b*) has shown, however, that the solution following the collision can be uniquely determined if certain conditions are imposed on the Ricci tensor. Feinstein, MacCallum and Senovilla (1989) have further pointed out that this ambiguity basically arises because the treatment of the physical problem is incomplete. In this case, the thin shell of matter is represented by a mathematical idealization in which the field equations for the matter are neglected.

When considering the collision of thin plane shells of matter, it is therefore inevitable that the ambiguities discussed above will also occur. In the above-mentioned solution of Dray and 't Hooft (1986) the ambiguity was effectively removed by imposing the condition that the two shells pass through each other and continue along the same null surfaces $u = \text{constant}$ and $v = \text{constant}$. If the two shells had been permitted to scatter each other, there would have been a totally different solution with non-zero Ricci tensor in the interior of region IV. A number of such solutions which include that of Dray and 't Hooft have been given by Taub (1988*a*).

A larger class of solutions describing the collision of thin plane shells of null matter followed by gravitational waves has been given been given by Tsoubelis (1989). These have been obtained simply by taking the Szekeres (1972) class of solutions and relaxing the boundary conditions to permit the case when $n_1 = n_2 = 1$, thus producing impulsive matter components along the boundaries of the different regions. Further solutions of this type can easily be obtained in the same way by taking any colliding plane gravitational wave solution and modifying the parameters to satisfy the relaxed boundary conditions.

It is also appropriate in this section to include a most interesting solution obtained by Babala (1987). This describes the collision of a plane impulsive gravitational wave with a plane shell of null matter and is of particular interest because the global structure of the solution has been described in detail. In this solution, the line element describing the space-time can be written in the simple form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(1 - u\Theta(u) + v\Theta(v)\big)^2\mathrm{d}x^2 - \big(1 - u\Theta(u) - v\Theta(v)\big)^2\mathrm{d}y^2. \tag{20.19}$$

It can thus be seen that

$$f = \tfrac{1}{2} - (2u + u^2)\Theta(u), \qquad g = \tfrac{1}{2} - v^2\Theta(v). \tag{20.20}$$

Since there is clearly a discontinuity in f' when $u = 0$, a thin shell of matter must exist on this hypersurface. There is also an impulsive gravitational wave on the hypersurface $v = 0$.

Remarkably, the solution in the interaction region IV is flat. The two impulsive waves simply pass through each other. The gravitational wave is focused by the thin shell of matter to a point $u = 1$, $v = 0$. However, the shell of null matter is focused astigmatically by the gravitational wave onto a line $u = 0$, $v = 1$ which has the topology of a circle. The plane gravitational wave is thus transformed after the collision into a converging spherical wave. The flat space-time after the collision does not extend to spatial infinity in all directions, but fills a cylinder of the same radius as the circular focus. The x coordinate in this case is periodic. Each period of x covers the cylinder once.

The space-time cannot be uniquely extended beyond the singularity $u + v = 1$ because of the topological singularities that occur when $u = 1$, $v = 0$ and $u = 0$, $v = 1$. It may also be noted that the global structure of this solution is remarkably similar to that of the Bell–Szekeres solution for colliding electromagnetic waves as analysed by Clarke and Hayward (1989).

A generalization of the solution of Babala has been given by Feinstein and Senovilla (1989). This describes the collision between a variably polarized plane gravitational wave and a shell of null matter. The line element has been taken in the form (11.4) and the solution has been obtained on the assumption that the metric function ω, which is the imaginary part of the Ernst function Z and describes the variation in polarization, is a function of v only. In the resulting class of solutions, the approaching gravitational wave with variable polarization can have an arbitrarily smooth wave front, and the opposing wave is a shell of null dust combined with a gravitational wave with constant linear polarization. A curvature singularity occurs on the focusing hypersurface.

20.5 Neutrino fields

Another interesting situation that was considered some time ago by the author (Griffiths 1976*a*), is the collision and interaction of two classical neutrino fields. The neutrino flux vector is necessarily null. It is therefore easy to set up the initial conditions in which two neutrino fields approach and collide. In fact, since plane electromagnetic waves can be re-interpreted in terms of neutrino fields, it is possible to consider initial metrics in regions I, II and III that are identical to those for colliding electromagnetic waves. The interaction following the collision, however,

is substantially different and indicates a number of very interesting properties.

A neutrino field is usually defined by a two-component spinor[1] ϕ_A which satisfies the classical neutrino Weyl equation

$$\sigma^{\mu}{}_{AB'}\phi^{A}{}_{;\mu} = 0, \qquad \text{or} \qquad \nabla_{AB'}\phi^{A} = 0 \tag{20.21}$$

and whose energy-momentum tensor is given by

$$T_{\mu\nu} = i\Big[\sigma_{\mu AB'}\big(\phi^{A}\phi^{B'}{}_{;\nu} - \phi^{B'}\phi^{A}{}_{;\nu}\big) + \sigma_{\nu AB'}\big(\phi^{A}\phi^{B'}{}_{;\mu} - \phi^{B'}\phi^{A}{}_{;\mu}\big)\Big]. \tag{20.22}$$

It is convenient to convert these forms from spinor notation to the Newman–Penrose formalism involving spin coefficients by first expanding the neutrino spinor in terms of two basis spinors o_A and ι_A as

$$\phi_A = \phi o_A + \psi \iota_A. \tag{20.23}$$

With this, the neutrino Weyl equation (20.24) can be expanded in the form

$$\begin{aligned} D\phi + \bar{\delta}\psi &= (\rho - \epsilon)\phi + (\alpha - \pi)\psi \\ \delta\phi + \Delta\psi &= -(\beta - \tau)\phi - (\mu - \gamma)\psi \end{aligned} \tag{20.24}$$

The Ricci scalar is zero,

$$\Lambda = 0, \tag{20.25}$$

and the remaining components of the Ricci tensor take the apparently complicated form

$$\begin{aligned} \Phi_{00} &= 8\pi i\left[\psi D\bar{\psi} - \bar{\psi}D\psi + \kappa\phi\bar{\psi} - \bar{\kappa}\psi\bar{\phi} + (\epsilon - \bar{\epsilon})\psi\bar{\psi}\right] \\ \Phi_{01} &= 4\pi i\big[\psi\delta\bar{\psi} - \bar{\psi}\delta\psi - \psi D\bar{\phi} + \bar{\phi}D\psi - \kappa\phi\bar{\phi} + \sigma\phi\bar{\psi} \\ &\qquad - (\bar{\rho} + \epsilon + \bar{\epsilon})\psi\bar{\phi} + (\beta - \bar{\alpha} - \bar{\pi})\psi\bar{\psi}\big] \\ \Phi_{02} &= -8\pi i\left[\psi\delta\bar{\phi} - \bar{\phi}\delta\psi + \sigma\phi\bar{\phi} + (\bar{\alpha} + \beta)\psi\bar{\phi} + \lambda\psi\bar{\psi}\right] \\ \Phi_{11} &= 4\pi i\big[\phi D\bar{\phi} - \bar{\phi}D\phi + \psi\Delta\bar{\psi} - \bar{\psi}\Delta\psi + (\bar{\epsilon} - \epsilon)\phi\bar{\phi} \\ &\qquad + (\tau + \bar{\pi})\phi\bar{\psi} - (\bar{\tau} + \pi)\psi\bar{\phi} + (\gamma - \bar{\gamma})\psi\bar{\psi}\big] \\ &= 4\pi i\big[-\phi\delta\bar{\psi} + \bar{\psi}\delta\phi - \psi\bar{\delta}\bar{\phi} + \bar{\phi}\bar{\delta}\psi + (\bar{\rho} - \rho)\phi\bar{\phi} \\ &\qquad + (\bar{\alpha} + \beta)\phi\bar{\psi} - (\alpha + \bar{\beta})\psi\bar{\phi} + (\mu - \bar{\mu})\psi\bar{\psi}\big] \\ \Phi_{12} &= 4\pi i\big[\phi\delta\bar{\phi} - \bar{\phi}\delta\phi - \psi\Delta\bar{\phi} + \bar{\phi}\Delta\psi + (\bar{\alpha} - \beta - \tau)\phi\bar{\phi} \\ &\qquad + \bar{\lambda}\phi\bar{\psi} - (\mu + \gamma + \bar{\gamma})\psi\bar{\phi} - \bar{\nu}\psi\bar{\psi}\big] \\ \Phi_{22} &= 8\pi i\left[\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi + (\bar{\gamma} - \gamma)\phi\bar{\phi} + \bar{\nu}\phi\bar{\psi} - \nu\psi\bar{\phi}\right]. \end{aligned} \tag{20.26}$$

[1] For an introduction to spinors see Bade and Jehle (1953), Penrose (1960) or Penrose and Rindler (1985), although a detailed knowledge of spinor methods is not required here.

If the approaching neutrino fields are not coupled with associated gravitational waves, then initial conditions can be set up such that the line elements in regions I, II and III are given in the forms

$$
\begin{aligned}
\text{Region I}: &\quad \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \\
\text{Region II}: &\quad \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(\tfrac{1}{2} + f(u)\big)\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) \\
\text{Region III}: &\quad \mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(\tfrac{1}{2} + g(v)\big)\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)
\end{aligned}
\tag{20.27}
$$

exactly as for colliding pure electromagnetic waves. In the notation of (20.23-26), $\psi = 0$ in regions I and II and $\phi = 0$ in regions I and III.

The situation is now well posed, and a unique solution will exist in the interaction region IV with initial data given by the functions $f(u)$ and $g(v)$ and the values of $\phi(u,v)$ and $\psi(u,v)$ on the initial hypersurfaces $u = 0$ and $v = 0$.

Such a solution can be found by first assuming that the two waves continue to follow null geodesic congruences. This assumption is only justified by the fact that it leads to an exact solution satisfying the required initial conditions.

With the above assumption the field equations imply that, since the approaching waves are hypersurface-orthogonal, they will remain so throughout the interaction region. It is thus possible to put

$$
\kappa = \nu = 0, \qquad \rho = \bar\rho, \qquad \mu = \bar\mu, \qquad \tau = \bar\pi = 2\beta = 2\bar\alpha \tag{20.28}
$$

as in (6.11). It is thus possible to adopt the tetrad (6.3) and, assuming global plane symmetry, the differential opperators are given by (6.10). The field equations are then given by (20.24) and by inserting (20.26) into (6.15).

It may first be observed that the Ricci tensor components Φ_{10} and Φ_{21} are now both non-zero. It follows that, in this case, the spin coefficient α cannot remain zero. Using the coordinate system described in Section 6.1, it is therefore not possible to make a transformation to put X^i and Y^i zero simultaneously. The metric in the interaction region IV for colliding neutrino fields therefore cannot be written as the product of two two-spaces, and the line element cannot be written in either of the forms (6.20) or (11.4) that have been considered in previous chapters.

It is also appropriate to note that, if there are no approaching gravitational waves with components Ψ_4 and Ψ_0, there is no obvious reason why the spin coefficients σ and λ should become non-zero in the interaction region. These terms always appear for colliding electromagnetic waves and, consequently, gravitational waves are always generated by the collision. Although a general collision of neutrino fields may generate

gravitational waves, this is not necessarily the case and a particular example has been given by the author (Griffiths 1976*a,b*) in which this does not occur.

This particular solution has been obtained by considering the case when it is possible to put $\epsilon = 0$ and $\gamma = 0$ everywhere, and in this case the scale functions $A(u, v)$ and $B(u, v)$ may both be taken to be unity.

In this situation a class of solutions has been obtained in which

$$\sigma = 0, \qquad \lambda = 0 \tag{20.29}$$

throughout the interaction region, so that the two neutrino fields continue to follow shear-free and twist-free null geodesic congruences. This leads to the interesting conclusion that, in direct contrast to the effects of colliding electromagnetic waves, transverse gravitational waves are not necessarily generated by colliding neutrino fields. Colliding neutrino fields mutually focus each other, but they do not necessarily induce shear.

In region IV these solutions are algebraically general, but have $\Psi_0 = 0$ and $\Psi_4 = 0$, so that the null vectors l^μ and n^μ are principal null vectors of the gravitational field in the interaction region. With the above conditions, the field equations (6.15*i,j*) now imply that

$$\alpha = -4\pi i\phi\bar{\psi}. \tag{20.30}$$

With (6.15*n,p*), it then follows that

$$\rho\mu = -4\alpha\bar{\alpha}, \qquad \Phi_{11} = -4\alpha\bar{\alpha}, \qquad \Psi_2 = -4\alpha\bar{\alpha} \tag{20.31}$$

and the remaining equations take the form

$$D\phi = \rho\phi - \alpha\psi \tag{20.32a}$$
$$\Delta\psi = -\mu\psi + \bar{\alpha}\psi \tag{20.32b}$$
$$\Delta Y^i - DX^i = -4(\alpha\xi^i + \bar{\alpha}\bar{\xi}^i) \tag{20.32c}$$
$$D\xi^i = \rho\xi^i \tag{20.32d}$$
$$\Delta\xi^i = -\mu\xi^i \tag{20.32e}$$
$$D\rho = \rho^2 + 8\pi i(\psi D\bar{\psi} - \bar{\psi}D\psi) \tag{20.32f}$$
$$D\mu = 2\rho\mu \tag{20.32g}$$
$$\Delta\rho = -2\rho\mu \tag{20.32h}$$
$$\Delta\mu = -\mu^2 - 8\pi i(\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi) \tag{20.32i}$$
$$\Psi_1 = 4\pi i(\bar{\phi}D\psi - \rho\psi\bar{\phi} - \bar{\alpha}\psi\bar{\psi}) \tag{20.32j}$$
$$\Psi_3 = 4\pi i(\bar{\psi}\Delta\phi + \mu\phi\bar{\psi} + \alpha\phi\bar{\phi}) \tag{20.32k}$$

where $D = \partial_v$ and $\Delta = \partial_u$.

An exact solution of these equations satisfying the required boundary conditions is then given by

$$\begin{aligned}
\rho &= -\frac{f'}{2(f+g)}, \qquad \mu = \frac{g'}{2(f+g)} \\
\alpha &= -\frac{i}{4}\frac{\sqrt{f'g'}}{(f+g)}\exp[\tfrac{1}{2}i\log g' - \tfrac{1}{2}i\log f'] \\
\xi^2 &= \frac{1}{\sqrt{2(f+g)}}, \qquad \xi^3 = \frac{i}{\sqrt{2(f+g)}} \\
X^2 + iX^3 &= 2\sqrt{2}\int_0^v \frac{\bar{\alpha}}{\sqrt{f+g}}\,\mathrm{d}v \\
Y^2 + iY^3 &= -2\sqrt{2}\int_0^u \frac{\bar{\alpha}}{\sqrt{f+g}}\,\mathrm{d}u
\end{aligned} \tag{20.33}$$

with the neutrino field given by

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2k}}\sqrt{\frac{f'}{(f+g)}}\exp[\tfrac{1}{4}i\log(f+g) - \tfrac{1}{2}i\log f'] \\
\psi &= \frac{1}{\sqrt{2k}}\sqrt{\frac{g'}{(f+g)}}\exp[\tfrac{1}{4}i\log(f+g) - \tfrac{1}{2}i\log g'].
\end{aligned} \tag{20.34}$$

The metric in the interaction region is now given by

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \tag{20.35}$$

where

$$\begin{aligned}
l_\mu &= \delta^0_\mu, \qquad n_\mu = \delta^1_\mu \\
m_\mu &= \sqrt{\frac{(f+g)}{2}}\left[(X^2+iX^3)\delta^0_\mu + (Y^2+iY^3)\delta^1_\mu - \delta^2_\mu - i\delta^3_\mu\right].
\end{aligned} \tag{20.36}$$

It may be noted that this solution has the usual curvature singularity on the space-like hypersurface $f+g=0$ on which the components of the curvature tensor are unbounded. Unlike the case for colliding electromagnetic waves, however, colliding neutrino fields in this case do not start to shear and transverse gravitational waves are not generated. On the other hand, these colliding neutrino fields may be considered to generate the longitudinal gravitational wave components Ψ_1 and Ψ_3 in addition to the usual coulomb component Ψ_2.

A further more general class of solutions, in which the approaching neutrino fields are also coupled to gravitational waves, has been obtained by Blazhenova-Mikulich and Sibgatullin (1982).

20.6 Other null fields

The collisions of a variety of other types of null field have also been considered. It is appropriate at this point to consider a number of these.

Gürses and Kalkanli (1989), for example, have considered the collision and subsequent interaction of N-Abelian gauge plane waves. These are effectively independent electromagnetic waves described by components ϕ^a_A each separately satisfying Maxwell's equations. In this case, the energy-momentum tensor is given by

$$\Phi_{AB} = \phi^a_A \phi^a_B. \tag{20.37}$$

The solutions considered by Gürses and Kalkanli all have the Φ_{02} term zero. This is the term which normally indirectly induces gravitational waves in the interaction region. With this restriction, it is possible to put $V = W = 0$ everywhere. In this case, the polarizations of the approaching waves are constant and aligned and, after the collision, the waves focus each other without astigmatism. The solution in the interaction region is generally of type D with only the Weyl component Ψ_2 non-zero. The appropriate boundary conditions have been given by Hayward (1989*c*) and, as expected, these imply that a space-like curvature singularity necessarily develops in the interaction region on the focusing hypersurface.

As pointed out by Hayward (1989*c*), the specific Gürses–Kalkanli solution does not satisfy the Einstein–Maxwell equations (6.21) and (6.22). The only non-zero components of the Ricci tensor in the interaction region are Φ_{00} and Φ_{22}. The solution thus effectively describes the collision of any null fields which do not interact directly apart from the gravitational interaction through Einstein's equations. For example, it includes a solution of the author (Griffiths 1976*b*) which represents the collision between an electromagnetic wave and a neutrino field. However, since $V = W = 0$ everywhere, the approaching null fields must be conformally flat and so are not coupled with independent or impulsive gravitational waves with components Ψ_0 or Ψ_4.

The collisions of some other types of field have previously been mentioned in Section 20.3. It was pointed out that, in agreement with the results of Taub (1990) for colliding gravitational waves with null dust, Ferrari and Ibañez (1989*a,b*) have obtained a four-parameter family of

solutions of Einstein's equations for which the source is a non-perfect fluid with an anisotropic distribution of pressure. In this case, they have also shown that the Klein–Gordon equation for a massless scalar field is satisfied in the interaction region, and that the scalar field satisfies the required junction conditions across the null boundaries. It therefore follows that this class of solutions can alternatively be interpreted in terms of the collision and interaction of massless scalar mesons.

In the same way, Feinstein, MacCallum and Senovilla (1989) have reconsidered the solutions of Chandrasekhar and Xanthopoulos (1985*b*) which contain a stiff fluid in region IV, and have shown that the Klein–Gordon equation is satisfied everywhere. These solutions can, therefore, also be reinterpreted as the collision of two massless scalar mesons which produce a minimally coupled massless scalar field in the interaction region.

These points had previously been referred to in Section 20.3, where it was also pointed out that two different solutions, one for a stiff fluid and the other for a superposition of two null fluids, can have the same prior extension. This lack of uniqueness is very interesting and has caused much confusion.[2] This ambiguity appears only for fluid solutions and is a consequence of the fact that there are no independent field equations. It is certainly not possible for fields of different types to be generated by colliding plane waves (Hayward 1990*a*).

The collision of various types of null dust in the absence of approaching gravitational waves has been further considered by Tsoubelis and Wang (1990), and a number of exact solutions having different interpretations have been described. One of these solutions illustrates the scattering of a null fluid or a scalar plane wave by a shell of null dust. Another generalizes a solution of the author (Griffiths 1976*b*) which describes the collision between an electromagnetic wave and a neutrino field.

These solutions have again been obtained by first solving the field equations in the interaction region under various assumptions concerning the Ricci tensor. It is only after the solutions have been obtained that their interpretations are considered in terms of the physical properties of the approaching waves. This clearly explains the lack of uniqueness described above. This approach is not equivalent to that with well-posed initial data.

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[2] Some readers will recall a heated debate on this point at the 12th International Conference on General Relativity and Gravitation held at Boulder, Colorado in 1989.

21

RELATED RESULTS

It is finally appropriate to review a series of results which are related to the analysis of exact solutions describing the collision of plane waves in a flat background, which is the main subject of this book. The results described in this chapter differ from those given in previous chapters in a variety of ways. A number of interesting results have been obtained using different approaches, by considering the background in region I to be non-flat, or by considering alternative gravitational theories.

21.1 Other wave interactions

In this book we have considered specifically the collision and subsequent interaction of plane waves in a flat background. Initial conditions have been set up to consider this situation. There is, however, a considerable volume of literature in which wave interactions are considered in very different contexts.[1] Some very brief comments on some particular situations are now appropriate.

It may first be observed that the Gowdy cosmologies (Gowdy 1971) which have plane symmetry can be considered to represent closed universes built from interacting gravitational waves.[2] The similarity between these solutions and those for colliding plane waves has already been noted in Sections 10.7 and 17.3. In accordance with a cosmological interpretation, these solutions start with an initial 'big bang' singularity from which the waves emerge. They thus include the time reverse of the interaction region for some colliding plane waves.

A related situation has been considered by Hayward (1990*b*), who has considered plane symmetric space-times which are asymptotically flat in the past. These space-times are thus effectively the time reverse of some Gowdy cosmologies. This situation includes the collision of plane waves in a flat background as considered in previous chapters, but also includes solutions that are nowhere flat. The field equations are clearly identical

[1] For a review of studies on the interaction of gravitational waves with matter and fields, and their possible detection by these means, see Grishchuk and Polnarev (1980) and Sibgatullin (1984).

[2] See for example Centrella and Matzner (1979), Moncrief (1981), Carmeli, Charach and Malin (1981), Carr and Verdaguer (1983), Ibañez and Verdaguer (1983), Adams *et al.* (1982), Feinstein and Charach (1986) and Feinstein (1987).

to those considered above, but the initial conditions are different. For any general class of solutions, instead of imposing the colliding wave conditions described in Chapter 7, Hayward has simply imposed the condition that the space-time is asymptotically flat in the past. In terms of the familiar null coordinates u and v, the initial data are effectively those considered to be specified by initial asymptotic waves at past null infinity as $u \to -\infty$ and $v \to -\infty$. Hayward has given a particular solution in which the asymptotic waves have bounded amplitude, and has shown that interacting waves of this type generically produce a future curvature singularity.

21.2 Collisions in non-flat backgrounds

Apart from in the last section, all the colliding plane wave solutions that have been obtained have been considered to occur in a flat Minkowski background. This constitutes a very severe restriction that limits the application of the results obtained to real physical situations.

The collision of plane gravitational waves in a non-flat background has been considered by Centrella and Matzner (1979, 1982), and by Centrella (1980). This work was motivated by a consideration of situations in cosmology which involve the interaction of gravitational waves. The assumption of plane symmetry is imposed, so that the background region can be taken to be described by the Kasner metric. The approaching gravitational waves are then assumed to have the same plane symmetry. This situation is well posed and the interaction following the collision is uniquely determined.

In Chapter 5, it was described how the collision of initially non-expanding gravitational waves results in the astigmatic focusing of each wave after the collision. Such focussing will also occur in this case. However, if the background space-time is assumed to be expanding, the approaching gravitational waves may also be assumed to be expanding. In this case, as the two waves pass through each other, their rates of expansion will be reduced through the introduction of shear, but the waves will not necessarily start to contract. It may thus intuitively be expected that expanding gravitational waves will continue to expand after a collision, although with a slightly reduced rate of expansion. In this case, a singularity will be expected to occur in the past prior to the collision, and a future singularity will not necessarily be expected.

These properties are all confirmed by the detailed calculations of Centrella and Matzner (1979, 1982) using a Green's-function analysis, and also by Centrella (1980) using direct numerical techniques. Initial conditions are set on a time-like slice through three initial regions with

different Kasner metrics being used in each region. Impulsive gravitational waves are generated from the joins in the initial data, and these subsequently collide and interact. Numerical integration indicates the expected properties. An explicit analytic solution, however, has not yet been obtained.

21.3 Solitons

Solitons occur in a number of physical theories. They arise as a feature of certain types of non-linear differential equations. Gravitational solitons appear quite naturally in the general theory of relativity, and their properties have been extensively investigated, particularly in a cosmological context.

Gravitational solitons are localized perturbations of the gravitational field which propagate on a homogeneous background. They have no dispersion. It has been suggested that their interaction in a cosmological context could play an important role in the process of isotropization in the early universe.

Mathematically, exact solutions describing solitonic behaviour can be generated using the inverse scattering technique of Belinskii and Zakharov (1978) from an initial 'seed' solution. This technique can be applied whenever the space-time has two commuting Killing vectors. Such solutions can always be interpreted as having plane symmetry. It may be recalled that the Ferrari–Ibañez solution for colliding gravitational waves that was described in Section 10.4 was originally obtained using this technique.

The collision and interaction of solitons in general relativity was initially investigated by Ibañez and Verdaguer (1983). In this paper, they took the homogeneous Kasner solution as the 'seed' metric. This describes an expanding vacuum cosmology with an initial singularity. In this case, the Belinskii–Zakharov inverse scattering technique generates pairs of solitons that are identified at the initial singularity and then propagate in opposite directions. By generating two such pairs at different locations, it is possible to consider the collision and interaction of two solitons, one coming from each pair.

The solution given by Ibañez and Verdaguer (1983) has a diagonal metric, and can therefore be considered as describing the collision and interaction of solitons with constant aligned polarization. In this solution, the line element is taken in the form of (10.59), namely:

$$\mathrm{d}s^2 = \frac{e^{-S}}{2\sqrt{\tilde{t}}}\left(\mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{z}^2\right) - \tilde{t}\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right). \tag{21.1}$$

The exact solution is given by

$$
\begin{aligned}
e^V &= \tilde{t}^{\delta}\sigma_1\sigma_2\\
e^{-S} &= \frac{2\tilde{t}^{(\delta^2/2-8)}{\sigma_1}^{\delta}{\sigma_2}^{\delta}}{H_1H_2(1-\sigma_1)^2(1-\sigma_2)^2}\left\{\left[(\sigma_1+\sigma_2)\tilde{t}^2-\frac{8\tilde{z}_1\tilde{z}_2\sigma_1\sigma_2}{(1+\sigma_1)(1+\sigma_2)}\right]^2\right.\\
&\qquad\left.-\frac{64{\omega_1}^2{\omega_2}^2{\sigma_1}^2{\sigma_2}^2}{(1-\sigma_1)^2(1-\sigma_2)^2}\right\}^2
\end{aligned}
\tag{21.2}
$$

where δ is the parameter of the seed Kasner metric and, for $i = 1, 2$,

$$
H_i = (1-\sigma_i)^2 + \frac{16{\omega_i}^2{\sigma_i}^2}{(1-\sigma_i)^2\tilde{t}^2}, \qquad \tilde{z}_i = \tilde{z}_i^{\circ} - \tilde{z}. \tag{21.3}
$$

The functions $\sigma_i(\tilde{t}, \tilde{z})$ are obtained from the complex 'pole trajectory' equation

$$
{\mu_i}^2 - 2(\tilde{z}_i - i\omega_i)\mu_i + \tilde{t}^2 = 0, \qquad \sigma_i = \mu_i\bar{\mu}_i/\tilde{t}^2 \tag{21.4}
$$

and the 'soliton' parameters $\tilde{z}_i^{\circ}$ and ω_i are arbitrary real constants which indicate the 'origins' and 'widths' of the two pairs of solitons.

When seeking to interpret this solution it may be noted that, at the initial time $\tilde{t} = 0$, one pair of solitons is localized at $\tilde{z}_1^{\circ}$ and the other pair at $\tilde{z}_2^{\circ}$. For small values of ${\omega_i}^2 \ll 1$ the solitons are very localized. By analysing the structure of the curvature tensor, it can be shown that these solitons are gravitational fields with both radiative and non-radiative components. They can be interpreted as quasiparticles which evolve towards pure gravitational waves.

In each pair, the solitons propagate in opposite $\tilde{z}$ directions. The approaching solitons collide and pass through each other, forming the same type of interaction region that we have been considering for colliding plane waves. In this case, however, the interaction region tends asymptotically towards the Kasner background. A singularity does not develop. This result is not surprising as the solitons collide in an expanding background and the singularity is in the past at $\tilde{t} = 0$. The null vectors tangent to the solitons continue to expand after the collision, but the expansion is reduced by the familiar 'focusing effect'.

An exact solution describing a similar situation in which a soliton collides with a plane gravitational wave has been obtained by Céspedes and Verdaguer (1987). In this case, the initial gravitational wave is taken to be one of the class of inhomogeneous non-diagonal solutions of Wainwright (1979) that can be interpreted as a gravitational wave pulse propagating on a Kasner background. It is appropriate to restrict the choice of the

arbitrary function of a null coordinate to start with so that the simplest solution for a gravitational wave on a Kasner background can be obtained. This is used as a seed solution in a soliton transformation with four complex pole trajectories. The resulting metric is very complicated, but can be interpreted by considering certain asymptotic regions.

As indicated above, this technique generates solutions in which a pair of solitons are created at the initial singularity and propagate in opposite directions. In this solution one of the solitons subsequently collides with the gravitational wave pulse of the seed solution. Céspedes and Verdaguer (1987) have analysed the asymptotic structure of this solution, and have shown that the two solitons ultimately differ due to the interaction of one with the pulse wave. The soliton and the gravitational wave both continue to expand after the collision and there is a singularity in the past, but mutual focusing effects can be seen. In particular, they have demonstrated that the gravitational pulse wave becomes stronger and more polarized after the collision.

The exact solutions so far described in this section have both been vacuum solutions. The above solutions of Ibañez and Verdaguer (1983) have been extended to soliton collisions in cosmologies with matter by Cruzgate *et al.* (1988). The technique they have developed starts with a solution of the Einstein field equations coupled to a massless scalar field in a five-dimensional space-time with three commuting Killing vectors. Then, using the inverse scattering method of Belinskii and Zakharov (1978) and a suitable Kaluza–Klein dimensional reduction procedure, they have obtained solutions of the four-dimensional Einstein equations for a baratropic perfect fluid source. This $4m$-soliton family of solutions with complex pole trajectories depends on two arbitrary parameters.

These solutions have an initial curvature singularity. The soliton collision is associated with a high degree of anisotropy and inhomogeneity, leading to the formation of voids and halos. At large times, due to the matter–soliton interactions, the space-time tends asymptotically towards the Friedmann–Robinson–Walker background. The expression for the energy-momentum tensor is very complicated and the conditions for realistic fluids impose such severe constraints on the parameters that no satisfactory explicit solutions have been presented. However, a number of the explicit solutions considered can be re-interpreted in terms of the Brans–Dicke theory.

21.4 Alternative gravitational theories

Throughout this work, the collision and interaction of plane waves has been considered only in terms of Einstein's general theory of relativity.

There are, however, many alternative theories of gravitation, and it may be of interest to investigate the collision and interaction of waves in these theories. So far very little work has been done along these lines. In fact the results of only one paper will be mentioned here.

Rosenbaum, Ryan, Urrutia and Matzner (1986) have obtained an exact solution describing the collision of plane waves in classical $N = 1$ supergravity. In contrast to the situation in general relativity, this solution is non-singular everywhere. The two waves do not mutually focus each other. Although the waves shear, the Grassman algebra requires that the square of the shear vanishes and so, according to equation (5.1a), the waves are not induced to focus.

There is a clear need for more work to be undertaken to investigate the collision and interaction of plane waves in other gravitational theories, including quantum supergravity, in which the work of Rosenbaum, Ryan, Urrutia and Matzner may appear as some sort of semiclassical limit.

21.5 Numerical techniques

The emphasis throughout this book has been on the derivation and analysis of exact solutions describing colliding plane waves. It must be pointed out, however, that numerical techniques for analysing this situation have also been developed.

For their work on wave interactions in plane symmetric cosmologies, Centrella and Matzner (1979) and Centrella (1980) have developed techniques for numerically solving the field equations given Cauchy data on an initial space-like hypersurface. Their techniques follow the Hamiltonian approach of a (3+1) splitting of space-time. Their method can deal with the impulsive gravitational waves that are generated from the joins in the initial data and their subsequent collision and interaction. Centrella and Matzner (1982) have specifically applied this method to colliding plane impulsive waves in an expanding background. In this case the difficulties which occur with the development of future singularities are avoided.

Powerful numerical methods for dealing with the the characteristic initial-value problem in general relativity have been developed by Stewart and Friedrich (1982). In this case constraint-free initial data are imposed on a space-like 2-surface and two null hypersurfaces containing it. This is clearly the type of the initial data for the colliding plane wave problem as discussed in this book.

The approach of Stewart and Friedrich is based on a (2+2) splitting of space-time using two null directions. It was subsequently applied by Corkill and Stewart (1983) to a number of situations in which there exist

two Killing vectors. In particular, they have applied it to the numerical evolution of the Schwarzschild space-time and to the collision and interaction of certain plane gravitational waves in a flat background.

Although this approach has clear advantages over the Cauchy problem, there are a number of inherent difficulties. These arise particularly because of the singularities that develop in these situations. It is therefore necessary to work with families of hypersurfaces that do not intersect the singularity. Corkill and Stewart have been able to devise methods by which it is possible to deal effectively with the evolution equations right up to the curvature singularity. The results they have obtained for colliding impulsive gravitational waves are entirely consistent with the Khan–Penrose solution described in Chapter 3.

22

CONCLUSIONS AND PROSPECTS

It is appropriate at the conclusion of this work to make some attempt to summarize the main results that have been obtained, and to point out some areas that still need further clarification.

22.1 General conclusions

The work that has been reviewed in the previous chapters has involved the study of colliding plane waves. Such waves, however, must be considered only as convenient mathematical idealizations. Real waves are never plane. At a large distance from the source, it may be considered that an initially spherical wavefront may become approximately plane. This is reasonable. In this context, however, the assumption of plane symmetry involves not only the assumption that the curvature of the wavefront becomes negligible, but also the assumption that the wave is infinite in extent. It is this latter feature that must be considered to limit the applicability of the results obtained to real physical processes.

The field equations for colliding plane waves were initially presented above in Chapter 6 following the early work of Szekeres (1970, 1972) and Khan and Penrose (1971). The first exact solutions to be published were obtained using these equations. This whole approach to the subject, however, was altered by the work of Chandrasekhar and Ferrari (1984) and Chandrasekhar and Xanthopoulos (1985*a*), who showed that the field equations could also be written in a form similar to those for stationary axisymmetric space-times. It has been shown in Chapters 11 and 16 that the main field equations are identical to the well known Ernst equations. Once the equations are written in this form, a variety of generation techniques can be applied to obtain large classes of further solutions. Many explicit solutions have now been published, and have been reviewed above. It is also clear how further solutions may be obtained.

Initial conditions have been set up such that the approaching gravitational waves in regions II and III are described in terms of the components Ψ_4 and Ψ_0 respectively. These components continue into the interaction region, though with a modified amplitude and possibly phase. There is, however, an additional interactive gravitational component that always appears in the interaction region IV. This is described by the component Ψ_2.

When electromagnetic waves collide, it appears that gravitational waves are always generated by the collision. These may be impulsive waves that occur along the boundaries of region IV only, or they may appear throughout the interaction region. However, they must always appear.

Any interpretation of the above results concerning the collision of plane waves must bear in mind the fact that the situation is highly idealized. Nevertheless, some general features concerning the interaction of waves in general relativity may be indicated. A similar situation is well known in cosmology. Here the general expansion and, particularly, the initial singularity of the Friedmann universes turn out to be general features of relativistic cosmologies, rather than particular consequences of the high degree of symmetry that is assumed.

In the case of colliding plane waves, the basic feature that has been substantially demonstrated above is the focusing effect of gravitational waves, and the slightly different focusing effects of waves of other types of matter. When two waves pass through each other, they will inevitably tend to focus each other. In the exact solutions that have been presented, the approaching waves are non-expanding. Thus, after the collision, the two waves will increasingly contract towards a focus. A wave crossing a gravitational wave will be focused astigmatically. Had the approaching waves been initially expanding, then it is reasonable to assume that, after the collision, their expansion would be slightly reduced. However, exact solutions describing such situations have not yet been obtained, although this general conclusion is supported by the work of Centrella and Matzner as described in Section 21.2.

When considering the collision of non-expanding plane waves, the waves after the collision will inevitably tend to a focus. This focus appears as a singularity in the space-time, and is one of the main features of the exact solutions that have been reviewed. This space-like singularity that appears in the interaction region is found to be either a scalar polynomial curvature singularity or, occasionally, a quasi-regular coordinate singularity forming a Killing–Cauchy horizon, according to the particular situation.

It has been argued above that, for arbitrary initial conditions, the singularity in the interaction region may normally be expected to be a curvature singularity, and therefore a boundary to the space-time. This appears to be the generic situation.

There is, however, a large class of exact solutions in which the curvature is bounded in the interaction region and the singularity appears to be merely a coordinate singularity. Such solutions include the degenerate Ferrari–Ibañez solution and the Chandrasekhar–Xanthopoulos solution

that are both of type D, the algebraically general Feinstein–Ibañez solution, and the equivalent electromagnetic solutions, including that of Bell and Szekeres. In these cases, the singularity can locally be removed by a coordinate transformation, and it appears to be possible to extend the space-time beyond this surface. However, any such extension must be non-unique. Further singularities also appear in these solutions, although they may be of a topological character. It has also been shown that these coordinate singularities are unstable with respect to at least one class of perturbations in the initial data. It follows that colliding plane waves will generically produce curvature singularities.

Boundaries to the space-time also occur in regions II and III. These prevent an observer from passing from the background region I, through region II or III, to region IV beyond what would then be a naked singularity. These boundaries are also quasi-regular coordinate singularities, and have loosely been described as 'fold singularities'. These are closely related to the known global properties of plane waves.

These general features have largely been inferred from a consideration of one or two exact solutions. There is clearly scope for much further work, and we now turn to review the possibilities that are open.

22.2 Prospects for further work

Doubtless many more exact solutions describing colliding plane waves will be obtained in the next few years. However, there seems little point in simply generating more solutions of the same type just with a number of additional free parameters.

What would be much more significant would be to find a practical way to determine the solution in the interaction region for an arbitrary set of initial conditions. This would describe the interaction between two arbitrarily specified approaching waves. This problem has recently been addressed in a series of papers by Hauser and Ernst (1989*a,b*, 1990) as reviewed in Chapter 14. Further work is clearly required to develop practical methods to solve this problem.

It is also of great importance to further analyse the global structure of colliding plane wave solutions. At present the structure of only a few solutions are known in detail, and it has been demonstrated that the curvature singularity that occurs in most colliding plane wave solutions is in fact 'generic'.

The structure of the Khan–Penrose solution has been thoroughly analysed by Matzner and Tipler (1984) as described in Section 8.2. The Bell–Szekeres solution has similarly been analysed by Clarke and Hayward (1989) and the degenerate Ferrari–Ibañez solution by Hayward (1989*a*).

The global structures of these particular cases are qualitatively different. A complete analysis has also been given of a solution of Babala (1987), but these are very much exceptions. A similarly thorough analysis of other classes of exact solutions would be most useful.

While commenting on global structure, it may also be mentioned that Hayward (1989*b*) has given an interesting interpretation of the time reverse of colliding plane wave solutions that contain Killing–Cauchy horizons rather than curvature singularities in terms of the snapping of cosmic strings.

Another topic that requires further attention is that of the solution-generation techniques that are appropriate for colliding plane waves. A large number of such techniques are already well known for stationary axisymmetric space-times. The relationships between these techniques have been investigated by Cosgrove (1980, 1982) and others. These techniques apply to all space-times that have two commuting Killing vectors. They therefore apply to stationary axisymmetric space-times, to cylindrically symmetric space-times and to colliding plane waves. In all these cases, the main field equations can be expressed as the same Ernst equation. However, the boundary conditions appropriate to each situation are significantly different. Generating techniques for stationary axisymmetric space-times have been adapted to space-times with two space-like commuting Killing vectors by Kitchingham (1984), but only in the context of cosmological solutions. It is particularly the significance of the difference in the boundary conditions and the physical significance of the different techniques that still requires further attention.

Indeed, it may be appropriate to consider further the relationship between exact solutions for stationary axisymmetric space-times, cylindrically symmetric space-times and colliding plane waves. Clearly some solutions can be interpreted in different ways, while others have only a single interpretation as they do not satisfy the boundary conditions for the alternative situations.

It has also been pointed out in Section 12.6 that it is possible to generate stationary axisymmetric solutions of Einstein's equations by using twistor methods to construct self-dual Yang–Mills fields with appropriate symmetries (Ward 1983, Woodhouse 1987). The relation between this approach and other solution-generating techniques has been considered by Woodhouse and Mason (1988). However, it still remains to be demonstrated how twistor techniques may be developed to generate colliding plane wave solutions.

As again emphasized at the beginning of this chapter, in this monograph we have only been considering highly idealized situations in which exactly plane waves collide in a flat Minkowski background. These ideal-

izations constitute severe restrictions. To achieve any physical relevance, it will be necessary to show how the results obtained here can be extended to more realistic waves with curved wave fronts, and to collisions in non-flat backgrounds.

The collision of waves in an expanding background has been considered by Centrella and Matzner and described in Section 21.2. However, no exact solutions have been obtained. Although appropriate numerical techniques have now been developed and give interesting results, it would be very useful to have a number of analytic solutions. It is to be hoped that techniques may be developed that would enable such solutions to be obtained and thoroughly analysed.

Of even greater importance would be the presentation of exact solutions describing the collision of waves that did not have complete plane symmetry. A description of the collision of the more general class of *pp*-waves, for example, would be most interesting. It would then be possible to consider the collision of waves of finite energy, and of finite extent. However, it seems as though we are still a long way from obtaining exact solutions of this type, or even of setting up the appropriate initial conditions. In this context it may be noted that an initial study of the focusing properties of some *pp*-waves has been presented by Ferrari, Pendenza and Veneziano (1988).

An alternative approach has been suggested by Yurtsever (1988*b,d*), who has considered the collision of almost-plane waves and has argued that such collisions result in black hole type singularities surrounded by horizons. He has proved (Yurtsever, 1988*d*, 1989*a*) that colliding waves that are exactly plane symmetric across a region of finite (large) transverse size but which fall off in an arbitrary way at larger transverse distances inevitably produce singularities that have the same local structure as those for exact plane wave collisions. However, the global singularity structure for such situations and the existence of horizons are topics that require further investigation.

The related problem of the collision, or close encounter, of high-speed black holes has been considered in a substantial paper by D'Eath (1978). In the limit as the black holes approach the speed of light, the incoming gravitational fields in this case are concentrated in two plane-fronted shock regions. Aichelburg and Sexl (1971) have shown that the line element for the limiting case of a black hole moving with the speed of light is given by

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + 4\mu\log(X^2+Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{22.1}$$

which is clearly a *pp*-wave of the form (4.1). It is thus of importance to consider the collision of two such waves, both for collisions and close

encounters in which the source is displaced from the origin of the X, Y coordinates.

It turns out to be possible, by considering how the far fields only are distorted and deflected by the collision, to estimate the amount of gravitational radiation that would be produced by such an encounter. The structure of the curved shocks after the collision has been analysed using perturbation methods. For certain values of the parameters, D'Eath has shown that a significant fraction of the collision energy can be radiated away as gravitational waves. A further analysis of this problem is most important.

Finally, it may be pointed out that all the calculations that have been described have involved classical theories. Exact solutions have been obtained of the classical theory of general relativity. However, in describing the collision of gravitational or electromagnetic waves, we are effectively considering macroscopic averages of graviton–graviton or photon–photon interactions. Ultimately, it will be necessary to consider quantum effects in such interactions. First steps in this direction have been taken by Lapedes (1977), who applied the Arnowitt–Deser–Misner quantization, and also by Yurtsever (1989*b*), who has considered linear quantum field theory on a Khan–Penrose colliding plane impulsive wave background. This seems to be a particularly important area for future research.

In conclusion, it should again be emphasized that the subject of this book, and the results reported in it, form only a single step in understanding the non-linearity that is inherent in Einstein's gravitational field equations.